TYPE II_1 FACTORS WITH ARBITRARY COUNTABLE ENDOMORPHISM GROUP

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ABSTRACT. In [I], Ioana introduced three new invariants of type II_1 factors: the one-sided fundamental group, the endomorphism semigroup and the set of right-finite bimodules. In [I], he does not provide many computations of these invariants. In particular, the question whether these invariants can be trivial is left open. We give an explicit example of a type II_1 factor for which all three invariants are trivial. More generally, for any countable left-cancellative semigroup G, we construct a type II_1 factor M whose endomorphism semigroup is precisely G.

INTRODUCTION AND OVERVIEW OF THE PAPER

In [I, section 10.(II)], Ioana introduced three new invariants of type II_1 factors, but he provided few concrete computations. Here we provide a large class of type II_1 factors where we can compute these invariants. The invariants in question are "one-sided versions" of three classical invariants. Let M be a type II_1 factor. Then the one-sided fundamental group $\mathcal{F}_s(M)$ is defined to be

$$\mathcal{F}_s(M) = \{t \in \mathbb{R}^*_+ \mid \text{ there is a normal injective } *\text{-homomorphism } \varphi : M \to M^t\}.$$

Observe that this set contains 1, and is closed under multiplication and under taking sums. In fact, it is closed under taking infinite sums. This implies that $\mathcal{F}_s(M) = \mathbb{R}^*_+$ whenever $(0,1) \cap \mathcal{F}_s(M) \neq \emptyset$. In particular, whenever the fundamental group of M is non-trivial, it follows that the one-sided fundamental group is all of $\mathbb{R}^*_+$. Similarly, it follows that $\mathcal{F}_s(\mathrm{L}(\mathbb{F}_n)) = \mathbb{R}^*_+$ for all $n \in \mathbb{N}$, because $\mathrm{L}(\mathbb{F}_n) \subset \mathrm{L}(\mathbb{F}_{n+1}) = \mathrm{L}(\mathbb{F}_n)^t$ where $t = \sqrt{\frac{n-1}{n}} < 1$. Our examples are all on the other side of the spectrum: they satisfy $\mathcal{F}_s(M) = \mathbb{N}$.

The second invariant that Ioana introduced is the one-sided version on the outer automorphism group. This is called the endomorphism semigroup $\mathrm{End}(M)$. It is the set of all normal injective $*$-homomorphisms $\varphi : M \to M$, and two such $*$-homomorphisms φ_1, φ_2 are identified if there is a unitary $u \in M$ such that $\varphi_1 = \mathrm{Ad}_u \circ \varphi_2$. This way, it is clear that $\mathrm{End}(M)$ is a unital semigroup. But it does not have to be left nor right cancellative. For example, $\mathrm{End}(R)$, where R is the hyperfinite II_1 factor, is neither left nor right cancellative. This is easy to see

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explicitly: remember that $R \cong R \otimes R \cong (R \otimes R) \rtimes (\mathbb{Z}/2)$, where $\mathbb{Z}/2$ acts outerly on $R \otimes R$ by swapping the components of the tensor product. Write $\varphi_1, \varphi_2 : R \to R \otimes R$ for the embeddings that are given by $\varphi_1(x) = x \otimes 1$ and $\varphi_2(x) = 1 \otimes x$. Denote by $\psi : R \otimes R \to (R \otimes R) \rtimes (\mathbb{Z}/2)$ the obvious embedding. Then we see that

$$\begin{aligned} &(\mathrm{id} \otimes \varphi_1) \circ \varphi_1 = (\mathrm{id} \otimes \varphi_2) \circ \varphi_1 : R \to R \otimes R \otimes R && \text{but } \mathrm{id} \otimes \varphi_1 \neq \mathrm{id} \otimes \varphi_2 \text{ in } \mathrm{End}(R) \\ &\psi \circ \varphi_1 = \psi \circ \varphi_2 \text{ in } \mathrm{End}(R) && \text{but } \varphi_1 \neq \varphi_2 \text{ in } \mathrm{End}(R). \end{aligned}$$

Even though in general $\mathrm{End}(M)$ does not have to be left cancellative, our examples will be. We show that every countable left-cancellative unital semigroup appears as the endomorphism semigroup of some type II_1 factor. In particular, we find a type II_1 factor with trivial endomorphism semigroup. This solves Ioanas question for an example of such a type II_1 factor.

In fact, we show even more. Ioana introduced a third invariant that contains both the one-sided fundamental group and the endomorphism semigroup. This is the set $\mathrm{RFBimod}(M)$ of all M-M bimodules H that have finite dimension as a right M-module, up to isomorphism of M-M bimodules. This set is closed under the Connes tensor product and under finite direct sums. It is even closed under infinite direct sums, provided that the dimensions (i.e. the right dimensions over M) form a convergent series.

This invariant contains the previous two invariants. The set of all right dimensions of M-M bimodules is precisely the one-sided fundamental group. Moreover, the sum and product in $\mathbb{R}_+^*$ correspond to the direct sum and the Connes tensor product. The endomorphism semigroup corresponds precisely to the set of all M-M bimodules with right dimension equal to 1, and the product in $\mathrm{End}(M)$ corresponds to the Connes tensor product in $\mathrm{RFBimod}(M)$. We give an example of a type II_1 factor for which $\mathrm{RFBimod}(M)$ is as small as possible, i.e. all right-finite M-M bimodules are trivial bimodules (direct sums of $\mathrm{L}^2(M)$).

The results in this paper are based on Popas deformation/rigidity theory. More precisely, we combine techniques and results from [P2, P3], [IPP], [PV4, PV6], [IPV] and [PV1, PV2] in order to reduce the computation of $\mathrm{End}(M)$ to a problem in ergodic theory.

The ergodic-theoretic problem is the following. Let $\Lambda \curvearrowright (Y, \nu)$ be an ergodic probability measure preserving (p.m.p.) action of a not necessarily countable group. Another probability measure preserving action $\Lambda \curvearrowright (Z, \eta)$ of the same group is said to be a factor of $\Lambda \curvearrowright (Y, \nu)$ if there is a p.m.p. quotient map $\Delta : Y \to Z$ such that $\Delta(\lambda y) = \lambda \Delta(y)$ for almost all $y \in Y$ and this for all $\lambda \in \Lambda$. The map Δ is called a factor map. We denote by $\mathrm{Factor}(\Lambda \curvearrowright (Y, \nu))$ the set of all factor maps from (Y, ν) to itself. Composition of factor maps defines a semigroup operation, and the identity map is the identity element for this operation. This way, we see that $\mathrm{Factor}(\Lambda \curvearrowright (Y, \nu))$ is a right-cancellative semigroup.

Given an ergodic action $\Lambda \curvearrowright (Y, \nu)$, we construct a type II_1 factor M_Y such that $\mathrm{End}(M_Y) = \mathrm{Factor}(\Lambda \curvearrowright (Y, \nu))^{\mathrm{op}}$. It is easy to see that every compact group G is $G = \mathrm{Factor}(G \curvearrowright (G, h))$

where h denotes the Haar measure on G. Hence G appears also as the endomorphism semigroup of some type II_1 factor M. In these cases, all endomorphisms of M are in fact isomorphisms. Every compact right-cancellative unital semigroup is automatically a group, so this observation covers whole compact right-cancellative case.

The discrete case is more interesting. Here we have examples of semigroups that are not groups. Already the semigroup of natural numbers with addition form such a semigroup. In section 6, we show that every countable right-cancellative semigroup appears as the factor semigroup of an ergodic p.m.p. action $\Lambda \curvearrowright (Y, \nu)$. Hence every countable left-cancellative semigroup appears as $\mathrm{End}(M)$ for some type II_1 factor M.

Given $\Lambda \curvearrowright (Y, \nu)$, we construct M_Y as follows. We can consider Λ (in fact, a quotient of Λ) as a subgroup of $\mathrm{Aut}_\nu(Y)$. Observe that $\mathrm{Factor}(\Lambda \curvearrowright (Y, \nu))$ only depends on the closure of Λ in the usual Polish topology on $\mathrm{Aut}_\nu(Y)$. Hence we can replace Λ by a countable dense subgroup without changing $\mathrm{Factor}(\Lambda \curvearrowright (Y, \nu))$. From now on we assume that Λ is countable. Let Γ_1 be a hyperbolic property (T) group with trivial endomorphism semigroup. Let $\Sigma \subset \Gamma_1$ be an amenable subgroup. Consider $\Gamma = \Gamma_1 *_\Sigma (\Sigma \times \Lambda)$. Let $\Gamma \curvearrowright I$ be an action of Γ on a countable set.

Let (X_0, μ_0) be an atomic probability space with unequal weights, and consider the generalized Bernoulli action $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^I$. Consider the obvious quotient map $\pi : \Gamma \to \Lambda$. Following [PV4, PV6], we define an action of Γ on $X \times Y$ by the formula $g(x, y) = (gx, \pi(g)y)$. Then we set $M_Y = \mathrm{L}^\infty(X \times Y) \rtimes \Gamma$.

Let us give an idea why $\mathrm{End}(M_Y) = \mathrm{Factor}(\Lambda \curvearrowright (Y, \nu))^{\mathrm{op}}$. One inclusion is easy. Given $\Delta \in \mathrm{Factor}(\Lambda \curvearrowright (Y, \nu))$, we define an embedding $\varphi_\Delta : M_Y \to M_Y$ by the formula $\varphi((a \otimes b)u_g) = (a \otimes \Delta_*(b))u_g$, where $\Delta_*(b) = b \circ \Delta$ for every function $b \in \mathrm{L}^\infty(Y)$. The application $\Delta \mapsto \varphi_\Delta$ embeds $\mathrm{Factor}(\Lambda \curvearrowright Y)^{\mathrm{op}}$ into $\mathrm{End}(M_Y)$.

Now, let $\varphi : M_Y \to M_Y$ be an endomorphism of M_Y. We want to show that $\varphi = \varphi_\Delta$ up unitary conjugacy. Denote $A = \mathrm{L}^\infty(X)$ and $B = \mathrm{L}^\infty(Y)$. Techiques from [P2, P3] show that $\varphi(B \rtimes \Gamma) \subset B \rtimes \Gamma$, up to a unitary. This result depends crucially on the fact the Γ_1 has property (T) while the action $\Gamma \curvearrowright X$ is a generalized Bernoulli action. Similarly, techniques from [IPP] show that $\varphi((A \otimes B) \rtimes \Gamma_1) \subset (A \otimes B) \rtimes \Gamma_1$ up to unitary conjugacy. In fact, we can assume that both unitaries are the same. This result uses the facts that Γ_1 has property (T) while Γ is an amalgamated free product. Because Γ_1 is hyperbolic, [PV1, PV2] shows that $\varphi(A)$ can not be in $B \rtimes \Gamma_1 = B \otimes \mathrm{L}(\Gamma_1)$. Then [IPV, theorem 5.1] shows that $C = \varphi(A)' \cap (A \otimes B) \rtimes \Gamma_1$ embeds into $A \otimes B$. Now we apply [IPV, theorem 6.1] to conclude that $\varphi(A \otimes B) \subset A \otimes B$. Moreover, the endomorphism $\varphi : B \otimes \mathrm{L}(\Gamma_1) \to B \otimes \mathrm{L}(\Gamma_1)$ is described in the following way. We can consider every element in $B \otimes \mathrm{L}(\Gamma_1)$ as a map from Y to $\mathrm{L}(\Gamma_1)$. There is a field of group endomorphisms $\delta_y : \Gamma_1 \to \Gamma_1$ ($y \in Y$) such that $\varphi(u_g)(y) = u_{\delta_y(g)}$. All of these are inner, so we can assume that they are all trivial. Now we know that $\varphi(A \otimes B) \subset A \otimes B$ and $\varphi(u_g) = u_g$ for all $g \in \Gamma_1$. For a good choice for the action $\Gamma \curvearrowright I$, a direct computation shows that in fact $\varphi = \varphi_\Delta$ for some factor map $\Delta \in \mathrm{Factor}(\Lambda \curvearrowright Y)$.

Of course, in the above idea of the proof, we have been ignoring a lot of technical conditions. A more precise statement and proof are given in 4. In section 1, we remind the reader of some well-known results that are crucial for this paper. In section 2, we extend [IPV, theorem 5.1 and 6.1] to our setting. Section 3 introduces two properties of groups that are crucial in the next section. There we show our main result, theorem 4.1. In order to apply that main theorem, we need to give an example of a group Γ and an action $\Gamma \curvearrowright I$ that satisfies the conditions of theorem 4.1. This is not very hard, but it is technical. Section 5 is devoted to such an example. Finally, in section 6, we show that all countable right-cancellative semigroups appear as $\mathrm{End}(M)$ for some type II_1 factor M.

1. Preliminaries and Notations

1.1. **Relatively weakly mixing actions.** Relative weak mixing plays a crucial role in the proof of theorem 2.2. This property was introduced by Furstenberg in [F] and Zimmer in [Z1, Z2], in the case of actions on probability spaces. In [P4], Popa generalized this to actions on von Neumann algebras.

Definition 1.1 (see [P4, lemma 2.10])**.** *Let $D \subset (B, \tau)$ be an inclusion of finite von Neumann algebras. Assume that a countable group Γ acts trace-preservingly on B and leaves D globally invariant. Denote the action by α. We say that Γ acts weakly mixingly on B relative to D if one of the following equivalent conditions holds.*

(1) *There exists a sequence of group elements $(g_n)_n$ in Γ such that*
$$\|\mathrm{E}_D(x\alpha_{g_n}(y))\|_2 \to 0 \qquad \text{for all } x \in B, y \in B \ominus D.$$
(2) *Every Γ-invariant positive element a with finite trace in the basic construction $\langle B, e_D\rangle$, must be $a \in e_D D e_D$.*
(3) *The action of Γ on $\mathrm{L}^2(B) \otimes_D \mathrm{L}^2(B)$ is ergodic relative to $\mathrm{L}^2(D)$. I.e. all Γ-invariant vectors $\xi \in \mathrm{L}^2(B) \otimes_D \mathrm{L}^2(B)$ are $\xi \in \mathrm{L}^2(D)$.*
(4) *The only right D-submodules of $\mathrm{L}^2(B)$ that have finite dimension over D and are Γ-invariant, are contained in $\mathrm{L}^2(B)$.*

proof of equivalence of these conditions. For a proof that conditions (1) and (2) are equivalent, we refer to [P4, lemma 2.10]. Equivalence of conditions (2) and (3) follows from the fact the $\mathrm{L}^2(\langle B, e_D\rangle) = \mathrm{L}^2(B) \otimes_D \mathrm{L}^2(B)$. Condition (2) and (4) are equivalent because the D-submodules of $\mathrm{L}^2(B)$ correspond 1-to-1 with the projections in $\langle B, e_D\rangle$. The dimension of the submodule is precisely the trace of the corresponding projection, and the submodule is Γ-invariant if and only if its corresponding projection is. □

In the case where B is abelian, we recover the classical definition of relative weak mixing:

Observation 1.2. *Let $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action and suppose that $p : (X, \mu) \to (Y, \nu)$ is a factor of this action. Then $\mathrm{L}^\infty(Y, \nu) \subset \mathrm{L}^\infty(X, \mu)$ is a Γ-invariant von Neumann subalgebra. Then the following are equivalent:*

(1) *$\Gamma \curvearrowright \mathrm{L}^\infty(X, \mu)$ is weakly mixing relative to $\mathrm{L}^\infty(Y, \nu)$, in the sense defined above.*

(2) *$\Gamma \curvearrowright (X,\mu)$ is weakly mixing relative to $p : X \to Y$, in the classical sense. I.e. the diagonal action of Γ on $X \times_Y X$ is ergodic.*

1.2. **Generalized co-induced actions.** We introduced generalized co-induced actions in [D]. Here we repeat the construction and generalize some properties.

Definition 1.3. *Let $\Lambda \curvearrowright I$ be an action of a countable group on a countable set. Let $\omega : \Lambda \times I \to \Lambda_0$ be a cocycle. Suppose that Λ_0 acts probability measure preservingly on (Y_0,ν_0). Define an action of Λ on $(Y,\nu) = (Y_0,\nu_0)^I$ by the formula $(gy)_i = \omega(g,g^{-1}i)y_{g^{-1}i}$. This action is called the generalized co-induced action of $\Lambda_0 \curvearrowright (Y_0,\nu_0)$, with respect to ω.*

Lemma 1.4. *Let $\Lambda \curvearrowright I$ be an action of a countable group on a countable set, and let $\omega : \Lambda \times I \to \Lambda_0$ be a cocycle. Suppose that $\Lambda_0 \curvearrowright (Y_0,\nu_0)$ is a probability measure preserving action. Consider the generalized co-induced action $\Lambda \curvearrowright (Y,\nu) = (Y_0,\nu_0)^I$.*

- *If all orbits of $\Lambda \curvearrowright I$ are infinite, then the generalized co-induced action $\Lambda \curvearrowright (Y,\nu)$ is weakly mixing.*
- *Suppose that $\Lambda \curvearrowright I$ and ω satisfy the following three conditions.*
  - *Λ acts transitively on I.*
  - *There exists an $i \in I$ such that (or equivalently, for all $i \in I$) ω maps the set $\mathrm{Stab}\{i\} \times \{i\}$ surjectively onto Λ_0*
  - *There exists an $i \in I$ such that (or equivalently, for all $i \in I$) the subgroup $S_i = \{g \in \Lambda \mid gi = i \text{ and } \omega(g,i) = e\}$ acts with infinite orbits on $I \setminus \{i\}$.*

  Then every measurable Λ-invariant map $f : Y \to Y$ is of the form $f(x)_i = f_0(x_i)$, where $f_0 : Y_0 \to Y_0$ is a measurable Λ_0-invariant map.

Proof. The first point follows in the same way as for generalized Bernoulli actions. See for example [PV3, proposition 2.3].

For the second point, it is clear that every map of the given form is measurable and Λ-invariant. On the other hand, let $f : Y \to Y$ be a Λ-invariant map. Fix $i \in I$ and consider the composition $f_i : Y \to Y_0$ of f with the quotient onto the i-th component of Y. Observe that f_i is S_i-invariant, because S_i does not act on the i-th component. But by the first point, S_i acts ergodically on $Y_0^{I-\{i\}}$. So we see that $f(x)_i = f_i(x) = f_0(x_i)$ for some measurable map $f_0 : Y_0 \to Y_0$, but only for the one $i \in I$ we fixed.

Moreover, observe that $f_0(\omega(g,i)x_0) = \omega(g,i)f_0(x_0)$ for all $g \in \mathrm{Stab}\{i\}$. Since ω maps $\mathrm{Stab}\, i \times \{i\}$ onto Λ_0, we find that f_0 is Λ_0-equivariant. Let $j \in I$ be another element. We find $g \in \Lambda$ such that $gj = i$. Then we compute that

$$f(x)_j = \omega(g^{-1},j)(gf(x))_i = \omega(g,i)^{-1}f_0((gx)_i) = f_0(x_j).$$

$\square$

2. Generalizing two results from [IPV]

Our main theorem depends on the results from sections 5 and 6 from [IPV]. But in fact, we need a slightly more general statement of these two results. The proof is mainly an application

of the direct integral decomposition of a von Neumann algebra. Nevertheless, we think it is worthwhile to give a careful statement and a short proof.

Theorem 2.1 (a version of [IPV, section 5]). *Let Γ act on a countable set I in such a way that there is a number $\kappa \in \mathbb{N}$ such that $\operatorname{Stab} \mathcal{F}$ is finite whenever $|\mathcal{F}| \geq \kappa$. Choose a standard probability space (X_0, μ_0). Suppose that (B, τ) is a finite type I von Neumann algebra. Write $A = \mathrm{L}^\infty(X_0^I)$ and consider the von Neumann algebra $M = (A \rtimes \Gamma) \otimes B$.*

Let $p \in \mathrm{L}(\Gamma) \otimes B$ be a projection. Let $D \subset pMp$ be an abelian subalgebra. Write $\mathcal{G}$ for the normalizer of D inside pMp. Denote the intersection $\mathcal{G}_0 = \mathcal{G} \cap \mathcal{U}(p(\mathrm{L}(\Gamma) \otimes B)p)$. Assume that

- *D does not embed into B inside M.*
- *$\mathcal{G}''$ does not embed into $(A \rtimes \operatorname{Stab}\{i\}) \otimes B$ inside M, for any $i \in I$.*
- *$\mathcal{G}''$ does not embed into $\mathrm{L}(\Gamma) \otimes B$ inside M.*
- *$\mathcal{G}_0''$ does not embed into $(\mathrm{L} \operatorname{Stab}\{i\}) \otimes B$ inside $\mathrm{L}(\Gamma) \otimes B$ for any $i \in I$.*

Then we get that $C = D' \cap pMp \prec^{\mathrm{f}}_M A \otimes B$.

Proof. Set $\widetilde{D} = \mathcal{Z}(C)$, and observe that $\widetilde{D}$ is still an abelian subalgebra of pMp that is still normalized by $\mathcal{G}$, and $D \subset \widetilde{D}$. Hence $\widetilde{D}$ still satisfies the four conditions above. But we also get that $p\,\mathcal{Z}(B) \subset \widetilde{D}$.

We write $\mathcal{Z}(B) = \mathrm{L}^\infty(Y, \nu)$ for some standard measure space (Y, ν). Then we can take the direct integral decomposition $B = \int_Y^\oplus B_y d\nu(y)$. Likewise we can decompose $p = \int_Y^\oplus p_y d\nu(y)$, where each $p_y \in \mathrm{L}(\Gamma) \otimes B_y$.

We decompose $\widetilde{D}$ and C into $\widetilde{D} = \int_Y^\oplus D_y d\nu(y)$ and $C = \int_Y^\oplus C_y d\nu(y)$ respectively. Denote $M_y = (A \rtimes \Gamma) \otimes B_y$, and observe that C_y is the relative commutant of D_y inside $p_y M_y p_y$. Each unitary u in $\mathcal{G}$ decomposes intro a direct integral of unitaries u_y, each of which normalizes D_y.

All in all, we see that the inclusion $D_y \subset p_y M_y p_y$ satisfies the conditions of [IPV, theorem 5.1]. We obtain that $C_y \prec^{\mathrm{f}}_{M_y} A \otimes B_y$ for almost all $y \in Y$. Hence it follows that $C \prec^{\mathrm{f}}_M A \otimes B$. $\square$

We also want to give a similar variant to [IPV, theorem 6.1] But for our main theorem, we need a slightly more general version: using the notations from [IPV, section 6], we can not assume that $(\mathrm{Ad}_{\gamma(s)})_{s \in \Lambda}$ acts weakly mixingly on $\mathcal{Z}(C)$. Instead, we can only assume that the action is weakly mixing relative to a discrete subalgebra $D \subset \mathcal{Z}(C)$. This is not a hard generalization, but we have to adapt the statement of the theorem slightly. In order to simplify the statement of theorem 2.2, we incorporate [IPV, corollary 6.2.1].

Theorem 2.2 (our variant of [IPV, theorem 6.1 and corollary 6.2.1]). *Let $\Gamma \curvearrowright (X, \mu)$ be a free, ergodic and p.m.p. action. Let (B, τ) be a finite type I von neumann algebra. Write $A = \mathrm{L}^\infty(X)$ and consider the von Neumann algebra $M = (A \rtimes \Gamma) \otimes B$. Let $p \in \mathrm{L}(\Gamma) \otimes B$ be a projection with finite trace.*

Let $C \subset pMp$ be a von Neumann subalgebra and suppose that $\gamma : \Lambda \to \mathcal{U}(p(\mathrm{L}(\Gamma) \otimes B)p) \cap \mathcal{N}_{pMp}(C)$ is a group morphism satisfying the following conditions.

- *Λ does not have any non-trivial finite dimensional unitary representations.*
- *$\gamma(\Lambda)''$ does not embed into any $\mathrm{L}(\mathrm{Centr}\{g\}) \otimes B$ for any $e \neq g \in \Gamma$.*
- *Consider the action of Λ on $\mathcal{Z}(C)$ by conjugating with $\gamma(\Lambda)$. We assume that this action is weakly mixing relative to $D = \mathcal{Z}(C) \cap p(\mathrm{L}(\Gamma) \otimes B)p$.*
- *$\mathcal{Z}(C)' \cap pMp = C$ and $C \prec^{\mathrm{f}} A \otimes B$.*

Then we have the following

- *a partial isometry $v \in \mathrm{L}(\Gamma) \otimes B \otimes \mathcal{B}(\mathbb{C}, \ell^2(\mathbb{N})) \otimes \mathcal{B}(\mathbb{C}, \ell^2(\mathbb{N}))$ with left support equal to p and with right support $q = v^*v$ inside $\widetilde{B} = B \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes \ell^\infty(\mathbb{N})$, and*
- *a group morphism $\delta : \Lambda \to \mathcal{G}$ where $\mathcal{G} \subset \mathcal{U}(\mathrm{L}(\Gamma) \otimes \widetilde{B})$ is the group*

$$\mathcal{G} = \left\{ \sum_{g \in \Gamma} p_g u_g \;\middle|\; p_g \in \mathcal{Z}(q\widetilde{B}q) \text{ are projections with } \sum_g p_g = q \right\},$$

such that

$$v^* C v = q(A \otimes \widetilde{B})q \text{ and } v^* \gamma(s) v = \delta(s) \text{ for all } s \in \Lambda.$$

Proof. Write $\mathcal{Z}(B) = \mathrm{L}^\infty(Y, \nu)$ for some measure space (Y, ν). Decompose B as the direct integral of $(B_y)_{y \in Y}$. Observe that $p\,\mathcal{Z}(B) \subset D \subset \mathcal{Z}(C)$. Hence we can decompose both D and C as a direct integral over Y of von Neumann algebras D_y and C_y. Write $M_y = (A \rtimes \Gamma) \otimes B_y$ and observe that M is the direct integral of the M_y. We decompose p as the direct integral of the projections p_y.

Observe that $C_y \prec^{\mathrm{f}}{}_{M_y} A \otimes B_y$ and that $\mathcal{Z}(C_y)' \cap p_y M_y p_y = C_y$ almost everywhere. Moreover, we can consider the $\gamma(s)$ as measurable maps $y \mapsto \gamma(s, y)$ from Y into the group of unitary elements of $p_y(\mathrm{L}(\Gamma) \otimes B_y)p_y$. We remark that the $\gamma(s, y)$ normalize C_y. Consider the action of Λ on $\mathcal{Z}(C_y)$ that is given by conjugation with $\gamma(s, y)$. This action is still weakly mixing relative to D_y.

But D_y embeds fully into $A \otimes B_y$, while D_y is contained in $\mathrm{L}(\Gamma) \otimes B_y$. Hence it fully embeds into B_y (see lemma 2.3 below). Since B_y is a finite type I factor, it follows that D_y is an abelian discrete von Neumann algebra. Since Λ does not have non-trivial finite dimensional unitary representations, it can not act trace-preservingly on such a von Neumann algebra, unless the action is trivial. So we find a countable number of projections $p_{y,n} \in D_y$ such that $\sum_n p_{n,y} = p_y$ and $D_y p_{n,y} = \mathbb{C}\, p_{n,y}$. In other words, Λ acts weakly mixingly on $\mathcal{Z}(C_y)p_{n,y}$.

We can now apply [IPV, theorem 6.1 and corollary 6.2.1] to the inclusion $C_y p_{n,y} \subset p_{n,y} M_y p_{n,y}$. We find a partial isometry $v_{n,y} \in \mathrm{L}(\Gamma) \otimes B_y \otimes \mathcal{B}(\mathbb{C}, \ell^2(\mathbb{N}))$ with left support equal to $p_{n,y}$ and with right support $q_{n,y} = v_{n,y}^* v_{n,y} \in B_y \otimes \mathcal{B}(\ell^2(\mathbb{N}))$, and such that

$$v_{n,y}^* C_y v_{n,y} = q_{n,y}(A \otimes B_y \otimes \mathcal{B}(\ell^2(\mathbb{N})))q_{n,y}.$$

Moreover, we find a group morphism $\delta_{y,n} : \Lambda \to \Gamma$ and a finite-dimensional unitary representation $\pi_{n,y} : \Lambda \to \mathcal{U}(p_{n,y}(B_y \otimes \mathcal{B}(\ell^2(\mathbb{N})))p_{n,y})$ such that

$$v_{n,y}^* \gamma(s, y) v_{n,y} = \pi_{n,y}(s) \otimes u_{\delta(s)} \text{ for all } s \in \Lambda.$$

But we assumed that Λ does not have any non-trivial finite dimensional unitary representations, so we see that $\pi_{n,y}(s) = p_{n,y}$.

In fact, reading the proof of [IPV, theorem 6.1] carefully, we see that we can do all this in such a way that the $v_{n,y}$ depend measurably on y. Hence we can consider the partial isometry

$$v = \sum_n \left(\int_Y^{\oplus} v_{n,y} d\nu(y) \right) \otimes e_{1,n} \in \widetilde{N}\, \mathrm{L}(\Gamma) \otimes B \otimes \mathcal{B}(\mathbb{C}, \ell^2(\mathbb{N})) \otimes \mathcal{B}(\mathbb{C}, \ell^2(\mathbb{N})).$$

This partial isometry has left support p and its right support is given by $q = \int_Y^{\oplus} \sum_n q_{n,y} \otimes e_{n,n} d\nu(y) \in \widetilde{B}$. A direct computation shows that

$$v^* C v = q(A \otimes \widetilde{B})q.$$

The measureable field $\delta(s, y, n) = \delta_{n,y}(s)$ of group morphisms satisfies the condition

$$v^* \gamma(s) v = q \sum_g \chi_{\{(y,n) | \delta(s,y,n) = g\}} \otimes u_g,$$

so we see that $v^* \gamma(s) v \in \mathcal{G}$ for all $s \in \Lambda$. □

Lemma 2.3. *Let A, B be finite von Neumann algebras with trace-preserving actions of a countable group Γ. Consider $M = (A \otimes B) \rtimes \Gamma$. Let $D \subset p(B \rtimes \Gamma)p$ be a von Neumann subalgebra of some corner of $B \rtimes \Gamma$. If D embeds into $A \otimes B$ inside M, then D already embeds into B inside $B \rtimes \Gamma$.*

Proof. Suppose that D did not embed into B inside $B \rtimes \Gamma$. By definition, we find a sequence $(v_n)_n$ of unitaries in D such that

$$\|\mathrm{E}_B(x v_n y)\|_2 \to 0 \text{ for all } x, y \in B \rtimes \Gamma.$$

We want to show that D does not embed into $A \otimes B$, inside M. So we want to show that

$$\|\mathrm{E}_{A \otimes B}(x v_n y)\|_2 \to 0 \text{ for all } x, y \in M.$$

By Kaplansky's density theorem, we can assume that $x = (a_1 \otimes b_1)u_g$ and $y = u_h(a_2 \otimes b_2)$ for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$ and $g, h \in \Gamma$. We compute that

$$\begin{aligned} \|\mathrm{E}_{A \otimes B}(x v_n y)\|_2 &= \|(a_1 \otimes b_1)\, \mathrm{E}_{A \otimes B}(u_g v_n u_h)(a_2 \otimes b_2)\|_2 \\ &\leq \|a_1\|\, \|a_2\|\, \|b_1\|\, \|b_2\|\, \|\mathrm{E}_B(u_g v_n u_h)\|_2 \to 0, \end{aligned}$$

as required. □

3. Anti-(T) groups and groups with small normalizers

It is well-known that a group Γ that has the Haagerup property does not contain an infinite property (T) subgroup. Slighly more generally, for any p.m.p. action $\Gamma \curvearrowright (X, \mu)$, we know that the corresponding group-measure space von Neumann algebra $\mathrm{L}^\infty(X) \rtimes \Gamma$ does not contain a diffuse von Neumann subalgebra with property (T), see [P1].

Definition 3.1 (anti-(T) group)**.** *We say that a group Γ is anti-(T) if for every trace preserving action $\Gamma \curvearrowright (A,\tau)$ on an amenable von Neumann algebra A and for every projection $p \in A \rtimes \Gamma$, the von Neumann algebra $p(A \rtimes \Gamma)p$ does not contain a diffuse von Neumann subalgebra with property (T).*

This definition differs from the notion of an anti-(T) group that was introduced in [HPV]. Every group that has the anti-(T) property in the [HPV]-sense is anti-(T) in our sense, but not the other way around. The advantage of our notion is that it is stable under arbitrary amalgamated free products.

Lemma 3.2. *If Γ_1, Γ_2 are anti-(T) groups, then $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is still anti-(T).*

Proof. Let $\Gamma \curvearrowright (A,\tau)$ be a trace-preserving action on an amenable von Neumann algebra. Let $p \in A \rtimes \Gamma$ be a projection and suppose that $Q \subset p(A \rtimes \Gamma)p$ is a property (T) subalgebra. By [IPP, theorem 5.1], we know that $Q \prec A \rtimes \Gamma_i$ for $i = 1$ or 2. In particular, there is a $*$-homomorphism $\theta : Q \rtimes q(A \rtimes \Gamma_i)^n q$ for some $n \in \mathbb{N}$ and $q \in (A \rtimes \Gamma_i)^n$. Since Γ_i was anti-(T), it follows that Q is not diffuse. □

In [OP], Ozawa and Popa show that the free groups have the following property: the normalizer of every diffuse abelian subalgebra $B \subset \mathrm{L}(\mathbb{F}_n)$ is amenable. This result was generalized later in [PV1]. There Popa and Vaes show that for every trace-preserving action $\mathbb{F}_n \curvearrowright (A,\tau)$ on a finite amenable von Neumann algebra and every diffuse abelian subalgebra $B \subset p(A \rtimes \mathbb{F}_n)p$ of a corner of the crosses product, we have the following dichotomy. Either B embeds into A or the normalizer of B is amenable. In [PV2], they show that this property holds for all hyperbolic groups.

In this section, we introduce a similar but weaker property of groups. The advantage of our property is that it is implied by the Haagerup property and that it is stable under taking amalgamated free products.

We first introduce what we mean when we say that an abelian subalgebra $B \subset M$ has a large normalizer. In the following, we denote by $\mathrm{D}_n(\mathbb{C})$ the algebra of diagonal $n \times n$ matrices with complex entries.

Definition 3.3. *Let $B \subset M$ be an abelian subalgebra of a finite von Neumann algebra M. We say that B has a large normalizer if $\mathcal{N}_M(B)$ contains a group $\mathcal{G}$ that generates a diffuse property (T) subalgebra of M.*

Definition 3.4 (groups with small normalizers)**.** *We say that a group Γ has small normalizers if, for every trace-preserving action $\Gamma \curvearrowright (A, \mathrm{Tr})$ on a finite amenable von Neumann algebra, and every projection $p \in M = A \rtimes \Gamma$, every diffuse abelian subalgebra $B \subset pMp$ with large normalizer, embeds into A inside M.*

It follows immediately from [PV2, theorem 3.1 and lemma 2.4] that the following groups have small normalizers.

- hyperbolic groups

- lattices in rank one simple Lie groups with finite center
- Sela's limit groups

Any anti-(T) group Γ has small normalizers, simply because no amplification of the crossed product $A \rtimes \Gamma$ can have a von Neumann subalgebra with property (T).

Theorem 3.5. *The amalgamated free product of groups with small normalizers over an amalgam that has the anti-(T) property still has small normalizers.*

Proof. Let Γ_1, Γ_2 be two groups with small normalizers, and let Σ be a common subgroup with the Haagerup property. Consider the amalgamated free product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$. Let Γ act trace preservingly on an amenable finite von Neumann algebra (A, τ). Denote $M = A \rtimes \Gamma$ and take a projection $p \in M$. Let $B \subset pMp$ be an abelian subalgebra with a large normalizer. Consider a subgroup $\mathcal{G} \subset \mathcal{N}_{pMp}(B)$ that generates a property (T) subalgebra in pMp. Denote by $N = B \vee \mathcal{G} \subset pMp$ the von Neumann subalgebra that is generated by B and $\mathcal{G}$.

Now we use techniques from [IPP]. But we use the versions as stated in [PV5], because these versions are more convenient. Consider the word-length deformation as defined in [IPP, section 2.3] (see also [PV5, section 5.1]): we define completely positive maps $m_\rho : M \to M$ by $m_\rho(au_g) = \rho^{|g|} a u_g$ where $|g|$ denotes the word-length of g and ρ is a real number between 0 and 1. These completely positive maps converge to the identity pointwise, as $\rho \to 1$.

Because $\mathcal{G}''$ has property (T), we know that m_ρ converges to the identity uniformly in $\|\cdot\|_2$ on $\mathcal{G}$. Observe that $\mathcal{G}''$ does not embed into $A \rtimes \Sigma$ because Σ is anti-(T). So [PV5, lemma 5.7] yields a real number $0 < \rho_0 < 1$ and a $\delta > 0$ such that

$$\tau(w^* m_{\rho_0}(w)) > \delta \text{ for all unitaries } w \in B.$$

By property (T) we find $\rho \geq \rho_0$ such that $\|v - m_\rho(v)\|_2 < \frac{1}{8}\delta^2$ for all $v \in \mathcal{G}$. Because $\tau(w^* m_{\rho_0}(w))$ increases with ρ_0, we can assume that $\rho = \rho_0$. Hence we can conclude that $\tau(v^* w^* m_{\rho_0}(wv)) > \frac{1}{2}\delta$ for all $v \in \mathcal{G}$ and $w \in \mathcal{U}(B)$.

Theorem [IPP, theorem 4.3] shows that N embeds into either $A \rtimes \Gamma_1$ or $A \rtimes \Gamma_2$. Without loss of generality, we can assume that N embeds into $A \rtimes \Gamma_1$. Hence we find a non-zero partial isometry $v \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes M$ and a $*$-homomorphism $\theta : N \to q(\mathrm{M}_n(\mathbb{C}) \otimes A \rtimes \Gamma_1)q$ such that $xv = v\theta(x)$ for all $x \in N$. We can assume that q is the support projection of $\mathrm{E}_{\mathrm{M}_n(\mathbb{C}) \otimes A}(v^* v)$. The subalgebra $\theta(B) \subset q((\mathrm{M}_n(\mathbb{C}) \otimes A) \rtimes \Gamma_1)q$ still has large normalizer. Because Γ_1 was assumed to have small normalizers, we see that $\theta(B)$ embeds into $\mathrm{M}_n(\mathbb{C}) \otimes A$ inside $(\mathrm{M}_n(\mathbb{C}) \otimes A) \rtimes \Gamma_1$. It follows that B embeds into A. □

4. Proof of the main result

We want to show that all semigroups of the form $\mathrm{Factor}(\Lambda \curvearrowright Y)^{\mathrm{op}}$ appear is $\mathrm{End}(M)$ for some type II$_1$ factor M. We can always replace Λ by a countable dense subgroup of $\Lambda \subset \mathrm{Aut}(Y, \nu)$ without changing $\mathrm{Factor}(\Lambda \curvearrowright Y)$. From now on we assume that Λ is countable. Lemma 6.3

in section 6 below shows that we can also assume that Λ is anti-(T) and that all cocycles $\omega : \Lambda \times Y \to K$ with values in a compact group are trivial.

Recall the construction of M from the introduction. Let $\Gamma = \Gamma_1 *_\Sigma (\Sigma \times \Lambda)$ be an amalgamated free product group. Take an action $\Gamma \curvearrowright I$ of Γ on a countable set I. Choose a purely atomic base space (X_0, μ_0) with unequal weights. Set $(X, \mu) = (X_0, \mu_0)^I$ and consider the generalized Bernoulli action $\Gamma \curvearrowright X$. Consider that canonical quotient morphism $\pi : \Gamma \to \Lambda$ and define a new action $\Gamma \curvearrowright X \times Y$ by the formula $g(x, y) = (gx, \pi(g)y)$ for all $g \in \Gamma$ and almost all $(x, y) \in X \times Y$.

Define $M = \mathrm{L}^\infty(X \times Y) \rtimes \Gamma$. Observe that every $\Delta \in \mathrm{Factor}(\Lambda \curvearrowright Y)$ defines an endomorphism $\varphi_\Delta : M \to M$ by the formula $\varphi_\Delta((a \otimes b)u_g) = (a \otimes \Delta_*(b))u_g$, where $\Delta_a st$ is defined by $\Delta_*(b)(y) = b(\Delta(y))$ for all $b \in \mathrm{L}^\infty(Y)$ and almost all $i \in Y$. It is easy to see that two such endomorphisms $\varphi_\Delta, \varphi_\eta$ are unitarily equivalent if an only if $\Delta = \eta$. The map $\Delta \mapsto \varphi_\Delta$ embeds $\mathrm{Factor}(\Lambda \curvearrowright Y)^{\mathrm{op}}$ into $\mathrm{End}(M)$.

We give a set of conditions on the group Γ_1 and the action $\Gamma \curvearrowright I$ that ensures that all endomorphisms $\varphi : M \to M$ are of the form φ_Δ for some $\Delta \in \mathrm{Factor}(\Lambda \curvearrowright Y)$. In fact, under the same conditions, we also find the all right-finite M-M bimodules are direct sums of bimodules of the form $H_\Delta = {}_{\varphi_\Delta(M)}\mathrm{L}^2(M)_M$. In particular, we see that $\mathcal{F}_s(M) = \mathbb{N}$.

In the next section, we give an explicit example of a group Γ_1 and an action $\Gamma \curvearrowright I$ that satisfy the conditions of theorem 4.1.

Theorem 4.1. *Let $\Lambda \curvearrowright (Y, \nu)$ be an ergodic probability measure preserving action. Assume that Λ is anti-(T) as in definition 3.1 and that all 1-cocycles $\omega : \Lambda \times Y \to K$ with values in compact groups are trivial.*

Let Γ_1 ba a countable group that satisfies the conditions $G_1, \dots, G_5$ below.

- G_1 *Γ_1 has the small normalizers property introduced above.*
- G_2 *Γ_1 does not have non-trivial finite dimensional unitary representations.*
- G_3 *all centralizers $\mathrm{Centr}_{\Gamma_1}\{g\} \subset \Gamma_1$ of finite-order elements $e \neq g \in \Gamma_1$ have the Haagerup property.*
- G_4 *Γ_1 contains a property (T) subgroup G.*
- G_5 *G contains a non-amenable subgroup G_0 such that the commensurator of G_0 and G generate all of Γ_1.*

Let $\Sigma \subset \Gamma_1$ be an amenable subgroup that is not virtually abelian. Consider the amalgamated free product $\Gamma = \Gamma_1 *_\Sigma (\Sigma \times \Lambda)$. Let $\Gamma \curvearrowright I$ be an action of Γ on a countable set, satisfying the conditions $A_1, \dots, A_5$ below.

- A_1 *Γ acts transitively on I*
- A_2 *All stabilizers $\mathrm{Stab}\{i\}$ have the Haagerup property and $\mathrm{Stab}\{i\} \cap \Gamma_1$ is abelian.*
- A_3 *the stabilizers $\mathrm{Stab}\{i, j\}$ of two-point sets are trivial.*
- A_4 *The only injective group morphisms $\theta : \Gamma_1 \to \Gamma_1$ that map each $\mathrm{Stab}_{\Gamma_1}\{i\}$ into some $\mathrm{Stab}_{\Gamma_1}\{j\}$ up to finite index are inner.*
- A_5 *there is $i_0 \in I$ such that $\mathrm{Stab}\{i_0\} \cap s\Gamma_1 s^{-1}$ is infinite for all $s \in \Lambda$.*

Let X_0 be an atomic probability space with unequal weights. Write $X = X_0^I$ and let Γ act on X by generalized Bernoulli action. Consider the natural quitient morphism $\pi : \Gamma \to \Lambda$ and define a probability measure preserving action $\Gamma \curvearrowright X \times Y$ by by the formula $g(x, y) = (gx, \pi(g)y)$. Denote $M = \mathrm{L}^\infty(X \times Y) \rtimes \Gamma$.

Then for every normal $*$-homomorphism $\varphi : M \to M$ from M into itself is of the form

$$\varphi((a \otimes b)u_g) = (a \otimes \Delta_*(b))u_g \text{ for some factor map } \Delta \in \mathrm{Factor}(\Lambda \curvearrowright Y),$$

up to unitary conjugacy.

Moreover, if H is a right-finite dimensional M-M bimodule, them H is (isomorphic to) a finite direct sum of M-M bimodules of the form ${}_{\varphi_\Delta(M)}\mathrm{L}^2(M)_M$ for factor maps $\Delta \in \mathrm{Factor}(\Lambda \curvearrowright Y)$.

Proof. Observe that the statement about endomorphisms follows immediately from the statement about right-finite bimodules. So let H be a right-finite bimodule of M. Then we know that H is of the form $H = {}_{\varphi(M)}p(\mathbb{C}^n \otimes \mathrm{L}^2(M))_M$ for some $*$-homomorphism $\varphi : M \to p(M \otimes \mathrm{M}_n(\mathbb{C}))p$ from M into some finite amplification $pM^n p$ of M.

We prove in 5 steps that $\mathrm{Tr}(p) = k$ for some natural number k and φ is of the form

$$\varphi((a \otimes b)u_g) = \sum_{i=1}^{k} e_{i,i} \otimes (a \otimes (\Delta_i)_*(b))u_g,$$

up to unitary conjugacy, for some factor maps $\Delta_1, \dots, \Delta_k \in \mathrm{Factor}(\Lambda \curvearrowright Y)$.

Throughout the proof we will use the following notations for various subalgebras of M.

- $A = \mathrm{L}^\infty(X)$ and $B = \mathrm{L}^\infty(Y)$.
- $M_1 = (A \otimes B) \rtimes \Gamma_1$ and $M_2 = (A \otimes B) \rtimes (\Sigma \times \Lambda)$.
- $M_{(i)} = (A \otimes B) \rtimes \mathrm{Stab}\{i\}$ for $i \in I$.
- $N = B \rtimes \Gamma$.

We will also combine the notations above. That way, we write $M_{1,(i)}$ for $M_1 \cap M_{(i)}$. Similarly, we write $N_1 = N \cap M_1$ and so on.

We use the notation $M^n = M \otimes \mathrm{M}_n(\mathbb{C})$ for the amplification of M by integer numbers. Similarly, we write $B^n = B \otimes \mathrm{M}_n(\mathbb{C})$ and $N^n = N \otimes \mathrm{M}_n(\mathbb{C})$. We denote the action of Γ on A by σ.

step 1: *We can assume that $p \in N_1^n$ and*

$$\varphi(M_1) \subset pM_1^n p \text{ and } \varphi(N) \subset pN^n p.$$

Since Γ_2 has the Haagerup property while G has property (T), we see that $\varphi(\mathrm{L}(G)) \not\prec_M M_2$. By [IPP, theorem 5.1], we find a partial isometry $v \in M \otimes \mathrm{M}_{n,m}(\mathbb{C})$ with left support $vv^* = p$ and with right support $q = v^*v \in M_1^m$, and such that $v^*\varphi(\mathrm{L}(G))v \subset qM_1^m q$. We conjugate φ by v and assume already that $p \in M_1^n$ and $\varphi(\mathrm{L}(G)) \subset pM_1^n p$.

Similarly, we see that $\varphi(\mathrm{L}(G))$ does not embed into $M_{(i)}$ for any $i \in I$. Applying [IPV, Corollary 4.3] (which is a version of [P2, theorem 4.1]) to the rigid inclusion $\varphi(\mathrm{L}(G)) \subset pM_1^n p$, we find a partial isometry $w \in M_1 \otimes \mathrm{M}_{n,k}(\mathbb{C})$ with left support $p = ww^*$ and with right support $r = w^* w \in N_1^k$, satisfying $w\varphi(\mathrm{L}(G))w^* \subset rN_1^k r$. We conjugate φ by w and we assume that $p \in N_1^n$ and that $\varphi(\mathrm{L}(G)) \subset pN_1^n p$.

Observe that $\varphi(\mathrm{L}(G_0))$ is contained in $pN_1^n p$, but it does not embed into P nor into $N_{(i),1}$ for any $i \in I$. So by [IPP, theorem 1.2.1] and [V2, lemma 4.2.1] (which is based on [P2, section 3]), it follows that its quasi-normalizer is still contained in $pN_1^n p$. But this quasi-normalizer contains $\varphi(B \otimes \mathrm{L}(\mathrm{Comm}_{\Gamma_1}(G_0)))$, and together with $\varphi(\mathrm{L}(G))$, this algebra generates $\varphi(N_1)$. We conclude that $\varphi(N_1) \subset pN_1^n p$.

We know that $\varphi(A)$ is an abelian subalgebra and all $\varphi(u_g) \in pM_1^n p$ with $g \in G$ normalize $\varphi(A)$. Moreover, $\varphi(\mathrm{L}(G))$ does not embed into P. It is shown in [IPP, theorem 1.4.1] that then $\varphi(A)$ itself is contained in $pM_1^n p$. Hence all of $\varphi(M_1)$ is contained in $pM_1^n p$.

We know that $\varphi(\mathrm{L}(\Lambda))$ commutes with $\varphi(\mathrm{L}(\Sigma)) \subset pN^n p$. Once we show that $\varphi(\mathrm{L}(\Sigma))$ does not embed into any $N_{(i)}$, inside N, then we can apply [V2, lemma 4.2.1] and conclude that $\varphi(\mathrm{L}(\Lambda))$ is contained in $pN^n p$. In that case, we find that $\varphi(N) \subset pN^n p$. It remains to show that $\varphi(\mathrm{L}(\Sigma))$ does not embed into $N_{(i)}$ for any $i \in I$. Observe that $N_{(i),1} = B \otimes \mathrm{L}(\mathrm{Stab}_{\Gamma_1}\{i\})$ is abelian, while Σ is not virtually abelian. It follows that $\varphi(\mathrm{L}(\Sigma))$ does not embed into $N_{(i),1}$. Hence (see [V2, remark 3.3]) we find a sequence $(v_m)_m$ of unitaries in $\varphi(\mathrm{L}(\Sigma))$ such that

$$\left\| \mathrm{E}_{N_{(i),1}}(x v_m y) \right\|_2 \to 0 \text{ for all } x, y \in N_1 \text{ and for all } i \in I.$$

We want to show that

$$\left\| \mathrm{E}_{N_{(i)}}(x v_m y) \right\|_2 \to 0 \text{ for all } x, y \in N \text{ and for all } i \in I.$$

By Kaplanski's density theorem, we can assume that $x = u_g, y = u_h$ with $g, h \in \Gamma$. Write the fourier expansion of v_m as $v_m = \sum_{k \in \Gamma_1} v_{k,m} u_k$. Then we compute that

$$\begin{aligned} \left\| \mathrm{E}_{N_{(i)}}(x v_m y) \right\|_2^2 &= \left\| \sum_{k \in \Gamma_1} \mathrm{E}_{N_{(i)}}(u_g v_{k,m} u_{kh}) \right\|_2^2 \\ &= \sum_{k \in \Gamma_1 \cap g^{-1} \mathrm{Stab}\{i\} h^{-1}} \| v_{k,m} \|_2^2 . \end{aligned}$$

If this last sum is non-empty, then there is a $k_0 \in \Gamma_1 \cap g^{-1} \mathrm{Stab}\{i\} h^{-1}$. Then it follows that $\Gamma_1 \cap g^{-1} \mathrm{Stab}\{i\} h^{-1} = \mathrm{Stab}_{\Gamma_1}\{g^{-1} i\} k_0$. So we see that

$$\begin{aligned} \left\| \mathrm{E}_{N_{(i)}}(x v_m y) \right\|_2^2 &= \sum_{k \in \mathrm{Stab}_{\Gamma_1}\{g^{-1} i\} k_0} \| v_{k,m} \|_2^2 \\ &= \left\| \mathrm{E}_{N_{(g^{-1} i),1}}(v_m u_{k_0}^*) \right\| \to 0. \end{aligned}$$

We have shown that $\varphi(\mathrm{L}(\Lambda))$ does not embed into $N_{(i)}$ for any i. It follows that $\varphi(N) \subset pN^n p$. This finishes the proof of our first step.

step 2: *We write $A_0 = \mathrm{L}^\infty(X_0)$ and for all subsets $J \subset I$, we denote by $A_0^J = \mathrm{L}^\infty(X_0^J)$ the subalgebra of A that consists of functions that depend only on the components indexed by J.*

Let $J \subset I$ be an infinite subset that is invariant under an infinite group $H \subset \Gamma_1$. Then we show that $\varphi(A_0^J)$ does not embed into B inside M_1, for any $i \in I$.

Observe that Γ_1 acts trivially on B, so B is contained in the center of M_1, in fact, it is the center of M_1. Remark that any given element $e \neq g \in \Gamma_1$ can fix at most one $j \in I$. So H acts freely on A_0^J, because J is infinite.

Now suppose that $\varphi(A_0^J)$ embeds into B inside M_1. Then we find a non-zero partial isometry $v \in M_1 \otimes \mathrm{M}_{n,m}(\mathbb{C})$ and a $*$-homomorphism $\theta : A_0^J \to qB^m q$, for some projection $q \in B^m$, such that $\varphi(x)v = v\theta(x)$ for all $x \in A_0^J$. But B^m is of finite type I and $\theta(A_0^J)$ is a (non-unital) abelian subalgebra. Up to a unitary in B^m, we can assume that $q \in B \otimes \mathrm{D}_m(\mathbb{C})$ and $\theta(A_0^J) \subset q(B \otimes \mathrm{D}_m(\mathbb{C}))q$ (see for example [V1, lemma C.2] for an argument). Taking a non-zero component of v, we can assume that $m = 1$ and hence that $v \in M_1 \otimes \mathrm{M}_{n,1}(\mathbb{C})$. Set $r = vv^* \in \varphi(A_0^J)' \cap pM_1^n p$ and observe that $\varphi(x)r = r(1 \otimes \theta(x))$ for all $x \in A_0^J$.

Set $w_g = \varphi(u_g)$ for all $g \in H$. We claim that r is orthogonal to $w_g r w_g^*$ for all $e \neq g \in H$. Fix $g \in H$. Since H acts freely on A_0^J, we find an element $a \in A_0^J$ such that $a - \sigma_g(a)$ is invertible. Then we compute that

$$\varphi(a)rw_g rw_g^* = r(1 \otimes \theta(a))w_g rw_g^* = rw_g(1 \otimes \theta(a))rw_g^* = rw_g r\varphi(a)w_g^* = rw_g rw_g^*\varphi(\sigma_g(a)).$$

But we know that $rw_g rw_g^*$ commutes with $\varphi(A_0^J)$, so $rw_g rw_g^*\varphi(a - \sigma_g(a)) = 0$. Since $a - \sigma_g(a)$ is invertible, it follows that r is orthogonal to $w_g rw_g^*$.

Since H is an infinite group, we have found an infinite sequence of pairwise orthogonal projections with the same trace in the finite von Neumann algebra $pM_1^n p$. This contradiction shows that $\varphi(A_0^J)$ can not embed into B inside M_1.

step 3: *From now on, we allow $n = \infty$. In that case, we denote $\mathrm{M}_\infty(\mathbb{C})$ for $\mathcal{B}(\ell^2(\mathbb{N}))$, and we write $\mathrm{M}_{k,\infty}(\mathbb{C})$ for $\mathcal{B}(\mathbb{C}^k, \ell^2(\mathbb{C}))$ and finally, we write $\mathrm{D}_\infty(\mathbb{C}) = \ell^\infty(\mathbb{N})$.*

With these notations we can assume that $p \in B^n$ is a projection with finite trace and that $\varphi(A \otimes B)$ is contained in $p(A \otimes B^n)p$. Moreover, we can assume that $\varphi(u_g) = u_g$ for all $g \in \Gamma_1$.

Denote $C = \varphi(A)' \cap pM_1^n p$, and observe that $M_1 = (A \rtimes \Gamma_1) \otimes B$, because Γ_1 acts trivially on Y. We want to apply theorem 2.1 to conclude that C embeds into $A \otimes B$ inside M_1. We check its four conditions. The first condition is satisfied by step 2 above. Observe that $\varphi(\mathrm{L}(G))$ has property (T) and hence can not embed into any of the amenable algebras $M_{(i),1}$ for any $i \in I$. This shows that the second and fourth condition are also satisfied.

If the third condition were not satisfied, then we had that $\varphi(A \rtimes G) \prec_{M_1} N_1$. So we find a partial isometry $0 \neq v \in M_1 \otimes \mathrm{M}_{n,m}(\mathbb{C})$ and a $*$-homomorphism $\theta : A \rtimes G \to qN_1^m q$, for some q, such that $\varphi(x)v = v\theta(x)$ for all $x \in A \rtimes G$. We can assume that q is the support projection of $\mathrm{E}_{N_1}(v^*v)$. Observe that $\theta(A)$ is an abelian subalgebra of $qN_1^m q$ with large normalizer. Since Γ_1 has small normalizers, we see that $\theta(A)$ embeds into B inside N_1. So there exists $0 \neq w \in N_1 \otimes \mathrm{M}_{m,k}(\mathbb{C})$ and a $*$-homomorphism $\rho : A \to rB^k r$ such that $\theta(x)w = w\rho(x)$. Since $vw \neq 0$, it follows that $\varphi(A)$ embeds into B inside M_1. But that is impossible by step 2 above. Hence also the third condition of theorem 2.1 is satisfied.

We conclude that $C \prec^{\mathrm{f}}_{M_1} A \otimes B$. We want to apply theorem 2.2 to the inclusion $C \subset pM_1^n p$. Denote by $\gamma : \Gamma_1 \to \mathcal{U}(pN_1^n p)$ the group morphism that is defined by $\gamma(g) = \varphi(u_g)$. Observe that Γ_1 does not have any non-trivial finite-dimensional representation and that $G \subset \Gamma_1$ is a property (T) subgroup. We show that $\gamma(\Gamma_1)''$ does not embed into $\mathrm{L}(\mathrm{Centr}_{\Gamma_1}\{g\}) \otimes B$ for any $e \neq g \in \Gamma_1$. If g is a finite-order element, then the centralizer has the Haagerup property, by assumption. If on the other hand g has infinite order, then we know that $\mathrm{L}(g^{\mathbb{Z}}) \otimes B$ is a diffuse abelian subalgebra of $\mathrm{L}(\Gamma_1) \otimes B$. If $\gamma(\Gamma_1)''$ embeds into the centralizer of g, then it follows that $\mathrm{L}(g^{\mathbb{Z}}) \otimes B$ has large normalizer. But Γ_1 is a group with small normalizers, so we conclude that $\mathrm{L}(g^{\mathbb{Z}}) \otimes B$ embeds into B, or still, that g has finite order. This contradicts our assumption. We conclude that $\gamma(\Gamma_1)''$ does not embed into $\mathrm{L}(\mathrm{Centr}_{\Gamma_1}\{g\}) \otimes B$ for any $e \neq g \in \Gamma_1$.

We show that the action by conjugation on $\mathcal{Z}(C)$ is weakly mixing relative to $D = \mathcal{Z}(C) \cap pN_1^n p$. Let $H \subset \mathrm{L}^2(\mathcal{Z}(C))$ be a finite dimensional, $\gamma(\Lambda_1)$-invariant right D-submodule. Observe that then $HpN_1^n p$ is a $\gamma(\Lambda_1)''$-$pN_1^n p$ subbimodule of $\mathrm{L}^2(pM_1^n p)$, and it has finite dimension on the right. Remark that $\gamma(\Gamma_1)''$ does not embed into $N_{(i),1}$ for any $i \in I$, so [V2, lemma 4.2.1] shows that any $\gamma(\Lambda_1)''$-$pN_1^n p$ subbimodule of $\mathrm{L}^2(pM_1^n p)$ that has finite dimension on the right, must be contained in $\mathrm{L}^2(pN_1^n p)$. It follows that H is contained in $\mathrm{L}^2(D)$.

We have just shown that $C \prec^{\mathrm{f}}_{M_1} A \otimes B$ and it is obvious from the definition of C that $\mathcal{Z}(C)' \cap pM_1^n p = C$, so we can apply theorem 2.2. This yields a partial isometry $v \in N_1 \otimes \mathcal{B}(\mathbb{C}^n, \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}))$ with left support $p = vv^*$ and with right support $q = v^*v \in \widetilde{B} = B \otimes \ell^\infty(\mathbb{N}) \otimes \mathcal{B}(\ell^2(\mathbb{N}))$, and such that

$$v^* C v = A \otimes \widetilde{B}.$$

Of course we still have that $v^*\varphi(N)v \subset q(N \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes \mathcal{B}(\ell^2(\mathbb{N})))q$.

Moreover, we find a group morphism δ from Γ_1 to the group

$$\mathcal{G} = \left\{ \sum_{g \in \Gamma_1} p_g u_g \;\middle|\; p_g \in B \otimes \ell^\infty(\mathbb{N}) \text{ are projections with } \sum_g p_g = q \right\}.$$

This group morphism satisfies $v^*\varphi(u_g)v = \delta(g)$ for all $g \in \Gamma_1$.

Denote by $Z \subset Y \times \mathbb{N}$ the support of q. Then we can view that group morphism $\Delta : \Gamma_1 \to \mathcal{G}$ as a measurable field $(\delta_z)_{z \in Z}$ of group morphisms $\delta_z : \Gamma_1 \to \Gamma_1$. We consider $\delta(g)$ as a map from Z to $\mathrm{L}(\Gamma_1)$, for all $g \in \Gamma_1$. As such, this map is given by $\delta(g)(z) = u_{\delta_z(g)}$.

We first show that almost all the δ_z with $z \in Z$ are injective. Observe that $A \rtimes \Gamma_1$ is a factor and that $\varphi(A \rtimes \Gamma_1)$ is contained in $(A \rtimes \Gamma_1) \otimes q\widetilde{B}q$. If not all δ_z were injective, then there is an element $g \in \Gamma_1$ and a non-null set $U \subset Z$ such that $\delta_z(g) = e$ for all $z \in U$. Denote by $r = \chi_U q$ the central projection in $(A \rtimes \Gamma_1) \otimes q\widetilde{B}q$ that corresponds to U. But then the $*$-homomorphism that maps $x \in A \rtimes \Gamma_1$ to $r\varphi(x)$ is not injective. This contradicts the factoriality of $A \rtimes \Gamma_1$.

Fix $i \in I$ and suppose that there is a non-null set $V \subset Z$ such that $\delta_z(\mathrm{Stab}_{\Gamma_1}\{i\}) \cap \mathrm{Stab}\{j\}$ does not have finite index in $\delta_z(\mathrm{Stab}_{\Gamma_1}\{i\})$, for any $j \in I$ and for all $z \in V$. In other words, $\delta_z(\mathrm{Stab}_{\Gamma_1}\{i\})$ acts with infinite orbits on I, for all $z \in V$. Denote by $r_2 = \chi_V q$ the central projection in $(A \rtimes \Gamma_1) \otimes q\widetilde{B}q$ that corresponds to $V \subset Z$. Then we see that $\varphi(A_0^i)r_2 \subset r_2\widetilde{B}r_2$. But we also get that $\varphi(A_0^{gi})r_2 = \varphi(u_g A_0^i u_g)r_2$ is contained in $r_2\widetilde{B}r_2$ for all $g \in \Gamma_1$. Hence we get that $\varphi(A_0^{\Gamma_1 i}) \subset r_2\widetilde{B}r_2$, but this contradicts step 2. So we can conclude that for almost all $z \in U$ and for every $i \in I$, a finite index subgroup of $\delta_z(\mathrm{Stab}_{\Gamma_1}\{i\})$ is contained in $\mathrm{Stab}_{\Gamma_1}\{j\}$ for some $j \in I$.

We know that all such injective group morphisms from Γ_1 to itself are inner. So each δ_z is inner. This is the same this as saying that δ itself is conjugate to $g \mapsto 1 \otimes u_g \in \mathcal{G}$, inside $\mathcal{G}$. So we find an element $u \in \mathcal{G}$ such that $u\varphi(u_g)u^* = u_g$ for all $g \in \Gamma_1$. We conjugate φ by u and assume that $\varphi(u_g) = u_g$. Remark that u normalizes $\widetilde{B}$, so we still have that

$$\varphi(A \otimes B) \subset A \otimes q\widetilde{B}q \subset A \otimes q(B \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes \mathcal{B}(\ell^2(\mathbb{N})))q \cong A \otimes qB^\infty q.$$

step 4: *We can assume that there is a cocycle $(b_s)_{s\in\Lambda}$ with values in $\mathcal{U}(pB^n p)$ such that $\varphi(u_g) = b_{\pi(g)}u_g$ for all $g \in \Gamma$. Moreover, we can assume that $\varphi(a) = a$ for all $a \in A$.*

Denote by $I_0 \subset I$ the set of all $i \in I$ such that $\mathrm{Stab}_{\Gamma_1}\{i\}$ is infinite. Observe that, for each $i \in I_0$, we have that $\varphi(A_0^{\{i\}})$ commutes with $\mathrm{L}(\mathrm{Stab}_{\Gamma_1})$. But $\mathrm{L}(\mathrm{Stab}_{\Gamma_1})$ does not embed into $B \rtimes \mathrm{Stab}\{i, j\}$ for any $j \neq i$, simply because $\mathrm{Stab}\{i, j\}$ is trivial. Now we can apply [V2, lemma 4.2.1] and we obtain that $\varphi(A_0^{\{i\}})$ is contained in $A_0^{\{i\}} \otimes pB^n p$.

We show that $\varphi|_{A_0^{\{i\}}}$ is in fact the identity morphism on $A_0^{\{i\}}$, for all $i \in I_0$. Take an element $g \in \Gamma_1$ such that $gi \neq i$. Observe that gi is still in I_0. Remember that $A_0 = \mathrm{L}^\infty(X_0, \mu_0)$ and that (X_0, μ_0) is a purely atomic probability space with unequal weights. Take a minimal projection q in A_0. Take a maximal abelian subalgebra $B_0 \subset pB^n p$ such that $q_1 = \varphi(q)$ and $q_2 = \varphi(\sigma_g(q)) = \sigma_g(q)$ are both in $A \otimes B_0$. We know that $B_0 \cong \mathrm{L}^\infty(Z, \eta)$ for some probability space (Z, η). We consider q_1 and q_2 as measurable maps from Z to A. Observe that $q_1(z)$ and $q_2(z)$ are independent for almost all $z \in Z$, i.e. $\tau(q_1(z)q_2(z)) = \tau(q_1(z))\tau(q_2(z)) = \tau(q_1(z))^2$. Write $f(z) = \tau(q_1(z))$ for all $z \in \widetilde{Z}$. Then we know that

$$\int_Z f(z)^2 dz = \tau(q_1 q_2) = \tau(q\sigma_g(q)) = \tau(q)^2 = \left(\int_Z f(z)dz\right)^2$$

The only positive functions satisfying this condition are the constant functions, so we see that $f(z) = \tau(q)$ almost everywhere.

We can do the same thing for all minimal projections $q_x = \chi^{\{i\}}_{\{x\}}$ in $A_0^{\{i\}}$. We find that $\varphi(q_x)(z)$ is a projection in $A_0^{\{i\}}$ with trace $\tau(\varphi(q_x)(z)) = \mu_0(\{x\})$, for almost every $z \in Z$ and for all $x \in X_0$. Moreover, we know that $1 = \sum_x q_x$, so we see that $1 = \sum_x \varphi(q_x)(z)$ for almost all $z \in Z$. Since μ_0 has unequal weights, it follows that $\varphi(q_x)(z) = q_x$ a.e, for all $x \in X_0$.

We prove that, for every $s \in \Lambda$, we get $\varphi(u_s) = b_s u_s$ for some unitary $b_s \in pB^n p$. Write $b_s = \varphi(u_s)u_s^* \in p(B^n \rtimes \Gamma)p$. By condition A_5, there is an element $i_0 \in I$ such that $\Lambda i_0 \subset I_0$. In particular, we see that Σi_0 and $s\Sigma i_0$ are contained in I_0. So b_s commutes with $A_0^{\Sigma i_0}$. We write the Fourier expansion of b_s as $b_s = \sum_{g\in\Gamma} b_{s,g}u_g$, where $b_{s,g} \in pB^n p$. Since b_s commutes with $A_0^{\sigma i_0}$, it follows that $a \otimes b_{s,g} = \sigma_g(a) \otimes b_{s,g}$ for all $g \in \Gamma$ and all $a \in A_0^{\Sigma i_0}$. If $b_{s,g}$ is nonzero, then we conclude that $a = \sigma_g(a)$ for all $a \in A_0^{\Sigma i_0}$. It follows that $g \in \mathrm{Stab}(\Sigma i_0)$. Since $\Sigma \subset \Gamma_1$ is not abelian, we know that Σ can not be contained in $\mathrm{Stab}_{\Gamma_1}\{i_0\}$. So Σi_0 contains at least two elements and hence $g \in \mathrm{Stab}(\Sigma i_0) = \{e\}$. We can conclude that $b_s \in pB^n p$, or still $\varphi(u_s) = b_s u_s$ with $b_s \in \mathcal{U}(pB^n p)$.

Since Γ_1 and Λ generate the group Γ, we see that $\varphi(u_g) = b_{\pi(g)}u_g$ for all $g \in \Gamma$. It remains to show that $\varphi(a) = a$ for all $a \in A$. We know already that $\varphi(a) = a$ for all $a \in A_0^{\{i_0\}}$. Since Γ acts transitively on I, the same holds for all $a \in A$.

step 5: *Conclude that theorem 4.1 holds.*

Consider $p \in B^n$ as a map from Y to $\mathrm{M}_n(\mathbb{C})$, or $\mathcal{B}(\ell^2(\mathbb{N}))$ if $n = \infty$. Then we see that $\mathrm{Tr}(p(y))$ is Λ-invariant. So by ergodicity of the Λ-action, it is constant, and up to conjugation by a unitary in B^n we can assume that p itself is constant, i.e. $p \in 1 \otimes \mathrm{M}_n(\mathbb{C})$. Reducing n we can assume that $p = 1$. Since p was a finite projection, it follows that n is now finite. Now we can consider $(b_s)_s$ as a cocycle for the action $\Lambda \curvearrowright Y$ with values in $\mathrm{U}_n(\mathbb{C})$. We assumed that all such cocycles are trivial, so up to conjugation with a unitary in B^n, we can assume that $b_s = 1$ for all $s \in \Lambda$.

Remark that $\varphi(B) \subset (A \otimes B^n) \cap (B^n \rtimes \Gamma) = B^n$ by steps 1 and 3. Since B is abelian, we can assume that $\varphi(B) \subset B \otimes \mathrm{D}_n(\mathbb{C})$. Any such $*$-homomorphism is given by a quotient map $\Delta : Y \times \{1, \dots, n\} \to Y$. This quotient map is Λ-equivariant because $\varphi(u_s) = u_s$ for all $s \in \Lambda$. The image $\Delta(Y \times \{k\})$ is Λ-invariant and non-null, so it must be all of Y up to measure 0. In other words, the formula $\Delta_k(y) = \Delta(y, k)$ defines a factor map for the action of Λ in Y. This works for all $k = 1, \dots, n$, so we see that our original right-finite bimodule H is a direct sum $H = \oplus_{k=1}^n H_\Delta$, finishing the proof of the theorem. $\square$

5. An example of a group action satisfying the conditions of theorem 4.1

In order to apply theorem 4.1, we have to find an action $\Gamma \curvearrowright I$ that satisfies the long list of conditions given there. Such examples are necessarily rather complicated. This section is devoted to the description of one such example.

As prescribed by theorem 4.1, the group Γ is of the form $\Gamma = \Gamma_1 *_\Sigma (\Sigma \times \Lambda)$. We build the group Γ_1 from an arithmetic lattice in $\mathrm{Sp}(n,1)$. We refer to [M] for an introduction to arithmetic lattices in Lie groups. For this section, we only need to know that suitable arithmetic subgroups are indeed lattices in the corresponding Lie groups.

Consider the set $\mathbb{H}\mathrm{ur}$ of Hurwitz quaternions, i.e.

$$\mathbb{H}\mathrm{ur} = \left\{ a+bi+cj+dk \;\middle|\; \text{either } a,b,c,d \in \mathbb{Z} \text{ or } a,b,c,d \in \mathbb{Z}+\frac{1}{2} \right\},$$

so the components are allowed to be either integers or half-integers, but mixtures are not allowed. This is a ring under the usual addition and multiplication of quaternions, so $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$. We denote the element $\frac{1}{2}(1+i+j+k)$ by h. The skew field of quaternions is denoted by $\mathbb{H}$.

The quaternions come with a natural involution defined by $\overline{a+bi+ci+dk} = a-bi-ci-dk$. This involution reverses the order of the multiplication, i.e. $\overline{xy} = \overline{x}\,\overline{y}$. Consider the sesquilinear form $B : \mathbb{H}\mathrm{ur}^3 \times \mathbb{H}\mathrm{ur}^3 \to \mathbb{H}\mathrm{ur}$ on $\mathbb{H}\mathrm{ur}^3$ that is defined by $B(\xi,\eta) = \overline{\xi_0}\eta_0 - \sum_{i=1}^2 \overline{\xi_i}\eta_i$. Observe that this form is of signature $(2,1)$. Consider the group

$$G = \mathrm{PSp}(B,\mathbb{H}\mathrm{ur}) = \{A \in \mathrm{M}_3(\mathbb{H}\mathrm{ur}) \mid B(A\xi,A\eta) = B(\xi,\eta) \text{ for all } \xi,\eta \in \mathbb{H}\mathrm{ur}^3\}/\{\pm 1\}.$$

This group is an arithmetic lattice in the Lie group $\mathrm{Sp}(2,1)/\{\pm 1\}$. As such it has property (T), and by [PV2], G has the small normalizers property.

We remark that $\mathrm{SL}_2\mathbb{Z}$ embeds into G, in the following way. Observe that $\mathrm{SL}_2\mathbb{Z}$ is exactly the set of matrices in $\mathrm{SL}_2\mathbb{Z}[i]$ that preserve the non-definite Hermitian form $B_0(\xi,\eta) = \overline{\xi_2}i\eta_1 - \overline{\xi_1}i\eta_2$. Consider the linear transformation $A : \mathbb{Z}[i]^2 \to \mathbb{H}\mathrm{ur}^2 \subset \mathbb{H}\mathrm{ur}^3$ that is defined by the matrix

$$A = \begin{pmatrix} h & \overline{h}i \\ i\overline{h} & -ihi \end{pmatrix}.$$

Observe that $B(A\xi,A\eta) = B_0(\xi,\eta)$, and that the matrix A is invertible over $\mathbb{H}\mathrm{ur}$. So A defines an embedding of $\mathrm{SL}_2\mathbb{Z}$ into G mapping a matrix $B \in \mathrm{SL}_2\mathbb{Z}$ to the block matrix

$$\begin{pmatrix} ABA^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

.

Consider the subgroup G_0 of all elements in $\mathrm{SL}_2\mathbb{Z}$ that are represented by matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{array}{l} \text{where } a,b,c,d \text{ are integers} \\ \text{and } b = 0 \mod 6, a = d = 1 \mod 6 \text{ and } c = 0 \mod 7. \end{array}$$

Define the group $\Gamma_1 = G *_{G_0} \mathrm{SL}_2\mathbb{Q}$, and consider its subgroup $\Sigma = \mathrm{ST}_2\mathbb{Q} \subset \mathrm{SL}_2\mathbb{Q}$ of upper triangular matrices in $\mathrm{SL}_2\mathbb{Q}$. Now it follows from theorem 3.5 that Γ_1 has the small normalizers property. The group $G \subset \Gamma_1$ has property (T). We also see that G_0 is almost normal in $\mathrm{SL}_2\mathbb{Q}$. But G and $\mathrm{SL}_2\mathbb{Q}$ together generate all of Γ_1. It was shown in [vNW] that $\mathrm{SL}_2\mathbb{Q}$ does not have any non-trivial finite-dimensional representations. Because the group G_0 generates all of G as a normal subgroup, the same is true for Γ_1. Moreover, the group Σ is amenable, but not virtually abelian.

We show that $\mathrm{Centr}_{\Gamma_1}\{g\}$ has the Haagerup property for every finite-order element $e \neq g \in \Gamma_1$. Observe that a finite-order element $e \neq g$ in Γ_1 is conjugate to an element in one of the components G or $\mathrm{SL}_2\mathbb{Q}$. We can assume without loss of generality that $g \in G$ or $g \in \mathrm{SL}_2\mathbb{Q}$. Moreover, G_0 is torsion-free, so g is not conjugate to an element in G_0. Hence the centralizer $\mathrm{Centr}_{\Gamma_1}\{g\}$ is still contained in the same component G respectively $\mathrm{SL}_2\mathbb{Q}$. If g were in $\mathrm{SL}_2\mathbb{Q}$, then this already implies that the centralizer of g has the Haagerup property. On the other hand, if g were in G, then the centralizer in Γ_1 is just the centralizer in G. This last centralizer is a discrete subgroup of the centralizer C of g in the Lie group $\mathrm{Sp}(2,1)/\{\pm 1\}$. The centralizer C in $\mathrm{Sp}(2,1)/\{\pm 1\}$ is a Lie group of strictly smaller dimension than $\mathrm{Sp}(2,1)/\{\pm 1\}$, hence its Lie algebra does not contain a copy of $\mathfrak{sp}(2,1)$ nor of $\mathfrak{sl}_2\,\mathbb{R} \ltimes \mathbb{R}^2$. Now it follows from [CCJ$^+$] or [dC] that C has the Haagerup property. The same is true for its discrete subgroup $\mathrm{Centr}_{\Gamma_1}\{g\}$.

Till now, we have checked all the conditions on the group Γ_1 that do not depend on the action $\Gamma \curvearrowright I$. We define the action $\Gamma \curvearrowright I$ as follows. Choose a one-to-one map $\Lambda \ni \lambda \mapsto n_\lambda \in \mathbb{N}$. For every $\Lambda \in \Lambda$, we set $x_\lambda = n_\lambda + i$. Consider the matrix

$$B_\lambda = \begin{pmatrix} \overline{x}_\lambda & 0 & n_\lambda \\ 0 & 1 & 0 \\ n_\lambda & 0 & x_\lambda \end{pmatrix} \in \mathbb{H}\mathrm{ur}^{3\times 3}\,.$$

This matrix defines an element in G. Consider likewise the element

$$C_\lambda = \begin{pmatrix} \frac{1}{n_\lambda} & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2\,\mathbb{Q}\,.$$

Finally, we define the element $h_\lambda = B_\lambda C_\lambda \in \Gamma_1$ and we consider the subgroup $H \subset \Gamma$ generated by the $\lambda h_\lambda \lambda^{-1}$. Finally, we define I to be the set of left cosets of H, i.e $I = \Gamma/H$, with the natural action of Γ by left translation.

Observe that all stabilizers of this action are conjugate to H, which in turn is isomorphic to $\mathbb{F}_\infty$. In particular, the stabilizers all have the Haagerup property. Moreover, the intersection of any conjugate of H with Γ_1 is either trivial or a copy of $\mathbb{Z}$. So the stabilizers $\mathrm{Stab}_{\Gamma_1}\{i\}$ are abelian. Remark also that $H = \mathrm{Stab}_\Gamma\{e\}$ has an infinite intersection with all $\lambda\Gamma_1\lambda^{-1}$, so $\mathrm{Stab}_{\Gamma_1}\{\lambda H\}$ is infinite for all $\lambda \in \Lambda$.

It remains to check that all stabilizers of two-point sets are trivial, and that injective group morphisms $\delta : \Gamma_1 \to \Gamma_1$ that map stabilizers into stabilizers up to finite index are inner. We prove a lemma that implies both facts.

Lemma 5.1. *Let $e \neq a, b \in H$ be elements of H, let g be an element of Γ and consider an injective group morphism $\delta : \Gamma_1 \to \Gamma_1$. Observe that δ defines a group morphism (we abuse the notation and keep using the letter δ) $\delta : H \to \Gamma$ by the formula $\delta(\lambda h_\lambda \lambda^{-1}) = \lambda\delta(h_\lambda)\lambda^{-1}$. If we have $a = g\delta(b)g^{-1}$, then it follows that δ is inner, say $\delta = \mathrm{Ad}_h$ and moreover $gh \in H$.*

Proof. We begin by studying the injective group morphisms $\delta : \Gamma_1 \to \Gamma_1$.

step 1: *Every injective group morphism $\delta : \Gamma_1 \to \Gamma_1$ is bijective and moreover it is given by $\delta = \mathrm{Ad}_h \circ (\mathrm{Ad}_E *_{G_0} \mathrm{Ad}_F)$ where $h \in \Gamma_1$ and either*

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and } E = \begin{pmatrix} V & 0 \\ 0 & u \end{pmatrix} \text{ with } V = A \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} A^{-1}$$
$$\text{or } F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } E = \begin{pmatrix} V & 0 \\ 0 & u \end{pmatrix} \text{ with } V = A \begin{pmatrix} vk & 0 \\ 0 & -vk \end{pmatrix} A^{-1}$$

where $u \in \mathcal{U} = \left\{ \pm\alpha, \frac{\pm\beta\pm\gamma}{\sqrt{2}}, \frac{\pm1\pm i\pm j\pm k}{2} \;\middle|\; \alpha, \beta, \gamma = 1, i, j, k \right\}$ *and* $v \in \mathcal{U} \cap \mathbb{C}$.

Let $\delta : \Gamma_1 \to \Gamma_1$ be an injective group morphism. Since $\delta(G)$ has property (T), it must be contained in a conjugate of one of the two components $G, \mathrm{SL}_2\mathbb{Q}$ of Γ_1. It can not be contained in a conjugate of $\mathrm{SL}_2\mathbb{Q}$ because that group has the Haagerup property. So we find $y_0 \in \Gamma_1$ such that $y_0\delta(G)y_0 \subset G$.

Consider Zariski the closure of G_1 of $y_0\delta(G)y_0^{-1}$ inside the Lie group $\mathrm{Sp}(2,1)/\{\pm1\}$. Then we know that G_1 is a Lie group. If G_1 were not equal to $\mathrm{Sp}(2,1)/\{\pm1\}$, then we know that its Lie algebra is also strictly smaller that $\mathfrak{sp}(2,1)$. So it does not contain a Lie subalgebra of the form $\mathfrak{sp}(2,1)$ nor of the form $\mathfrak{sl}_2\mathbb{R} \ltimes \mathbb{R}^2$. By [CCJ+, dC], it follows that G_1 has the Haagerup property, which is absurd because it contains the discrete property (T) group $y_0\delta(G)y_0^{-1}$. So we conclude that $G_1 = \mathrm{Sp}(2,1)/\{\pm1\}$. Now the Margulis superrigidity theorem (or better said, its version for $\mathrm{Sp}(n,m)$, see [C]) shows that $\mathrm{Ad}_{y_0} \circ \delta|_G$ extends to an isomorphism of $\mathrm{Sp}(2,1)/\{\pm1\}$. All such isomorphisms are inner, so we find an element y in the product $(\mathrm{Sp}(2,1)/\{\pm1\})\,\Gamma_1$ such that $\delta(x) = y^{-1}xy$ for all $x \in G$.

On the other hand, we know that the element

$$1 + e_{1,2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2\mathbb{Q}$$

has roots of all order. The only elements in Γ_1 that have roots of all orders are conjugate to $1 + re_{1,2}$ for some $r \in \mathbb{Q}$. Hence we find an element $z \in \mathrm{GL}_2\mathbb{Q}\,\Gamma_1$ such that $\delta(1 + e_{1,2}) = z^{-1}(1 + e_{1,2})z$. Write

$$x_v = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \in \mathrm{SL}_2\mathbb{Q} \text{ with } v \in \mathbb{R}\setminus 0.$$

Then we know that $x_v(1 + e_{1,2})x_v^{-1} = 1 + v^2 e_{1,2}$. It follows that $\delta(x_v) = z^{-1}x_v z$ and then it also follows that $\delta(1 + e_{2,1}) = z^{-1}(1 + e_{2,1})z$. Since the $1 + re_{1,2}$ and $1 + re_{2,1}$ generate $\mathrm{SL}_2\mathbb{Q}$, it follows that $\delta(x) = z^{-1}xz$ for all $x \in \mathrm{SL}_2\mathbb{Q}$.

For every element $x \in G_0$, we must have $\delta(x) = y^{-1}xy$ and $\delta(x) = z^{-1}xz$. It follows that zy^{-1} commutes with G_0. The only elements in $\mathrm{GL}_2\mathbb{Q}\,\Gamma_1(\mathrm{Sp}(1,2)/\{\pm1\})$ that commute with G_0 are of the form EF^{-1} where E and F are of the form described earlier. It follows that δ is indeed of the required form.

Form now on, we assume that δ is of the form $\mathrm{Ad}_E * \mathrm{Ad}_F$ where E and F are as in step 1.

step 2: *Replacing g by an element in $Hg\delta(H)$ and adapting a, b accordingly, we can assume that g is of the form $\lambda g_1 \mu^{-1}$, while $a = \lambda h_\lambda^s \lambda^{-1}$ and $b = \mu\delta(h_\mu)^t\mu^{-1}$ for some $g_1 \in \Gamma_1$, some $\lambda, \mu \in \Lambda$ and $s, t \in \mathbb{Z}$.*

Among the elements in $Hg\delta(H)$, we can assume that g has minimal length. Write $g = \lambda_0 g_1 \lambda_1 \dots g_n \lambda_n$ where $\lambda_0, \lambda_1 \in \Lambda$, $\lambda_1, \dots, \lambda_{n-1} \in \Lambda - \{e\}$ and $g_1, \dots, g_n \in \Gamma_1 \setminus \Sigma$. Write b as a reduced word $b = \mu_1 b_1 \mu_1^{-1} \mu_2 b_2 \mu_2^{-1} \dots \mu_m b_m \mu_m^{-1}$ where $b_i \in h_{\mu_i}^{\mathbb{Z}}$ for all i. Then we study 5 cases.

case 1: $n \geq 2$, $m \geq 2$.
In that case, we know that $g_1 \notin h_{\lambda_0}^{\mathbb{Z}}\Sigma$ and that $g_n \notin \Sigma\delta(h_{\lambda_n^{-1}})^{\mathbb{Z}}$. The word

$$g\delta(b)g^{-1} = \lambda_0 g_1 \lambda_1 \dots g_n \lambda_n \mu_1 \delta(b_1)\mu_1^{-1} \dots \mu_m \delta(b_m)\mu_m^{-1}\lambda_n^{-1} g_n^{-1} \dots g_1^{-1}\lambda_0^{-1}$$

is reduced, except that maybe $\lambda_n = \mu_1^{-1}$ or $\lambda_n = \mu_m^{-1}$. But even then, we know that $g_n\delta(b_1)$ and $\delta(b_m)g_n^{-1}$ can not be contained in Σ. In any case, we get a reduced word for $g\delta(b)g^{-1}$ that starts with something of the form $\lambda_0 g_1$ where $g_1 \notin h_{\lambda_0}^{\mathbb{Z}}\Sigma$. Such an element can never be contained in H.

case 2: $n = 1$, $m \geq 2$ and $g_1 \notin h_{\lambda_0}^{\mathbb{Z}}\Sigma\delta(h_{\lambda_1})^{\mathbb{Z}}$
In this case, the argument is the same as for the first case: the obvious word for $g\delta(b)g^{-1}$ is almost reduced, and it can never be contained in H.

case 3: $n = 1$ and $g_1 \in h_{\lambda_0}^{\mathbb{Z}}\Sigma\delta(h_{\lambda_1})^{\mathbb{Z}}$.
We can assume that $g_1 \in \Sigma$, so g is of the form λg_1. For $g\delta(b)g^{-1}$ to be in H, we need at least that $\delta(b_m)g_1^{-1} \in \Sigma h_{\lambda\mu_m}^{\mathbb{Z}}$. A direct computation shows that this is only possible if $g_1 = e$. So we have that $\delta(b_m) = k_1 h_{\lambda\mu_1}^s$ for some $k_1 \in \Sigma$ and $s \in \mathbb{Z}$. But then it follows that $\delta(b_{m-1})k_1^{-1}$ is contained in $\Sigma h_{\lambda\mu_{m-1}}^{\mathbb{Z}}$. As before, it follows that $k_1 = e$. By induction, we see that $\delta(b_i) \in h_{\lambda\mu_i}^{\mathbb{Z}}$ for all i. Replacing b by any $\mu_i b_i \mu_i^{-1}$, we can assume that b is of the required form. The conjugating element g is already of the required form, and a is then automatically of the right form.

case 4: $n \geq 2$, $m = 1$
In this case, we can assume that $g_1 \notin h_{\lambda_0}^{\mathbb{Z}}\Sigma$. We get the following word for $g\delta(b)g^{-1}$:

$$g\delta(b)g^{-1}\lambda_0 g_1 \dots \lambda_{n-1} g_n \lambda_n \mu_1 \delta(b_1)\mu^{-1}\lambda_n^{-1} g_n^{-1}\lambda_{n-1}^{-1} \dots g_1^{-1}\lambda_0^{-1}.$$

If λ_n is not μ_1^{-1}, this word is already reduced. On the other hand, form the definition of H and Σ, it is clear that no element of the form $\delta(h_\lambda)^s$ can be conjugated into Σ, inside Γ_1. So if $\lambda_n\mu_1$ were e, then we still had that

$$g\delta(b)g^{-1}\lambda_0 g_1 \dots \lambda_{n-1}(g_n\delta(b_1)g_n^{-1})\lambda_{n-1}^{-1} \dots g_1^{-1}\lambda_0^{-1}$$

is a reduced word. In both cases, it is clear that $g\delta(b)g^{-1}$ can not be contained in H.

case 5: $n = 1$, $m = 1$ and $g_1 \notin h_{\lambda_0}^{\mathbb{Z}} \Sigma \delta(h_{\lambda_1})^{\mathbb{Z}}$
Now we see that

$$a = g\delta(b)g^{-1} = \lambda_0 g_1 \lambda_1 \mu_1 \delta(b_1)\mu_1^{-1}\lambda_1^{-1} g_1^{-1}\lambda_0^{-1}.$$

If $\lambda_1\mu_1 \neq e$, then this is a reduced expression for an element that is not in H. Otherwise, a, b and g are of the required form. This finishes the proof of step 2.

step 3: *It follows that* $\delta = \mathrm{id}$, $s = t$, $\mu = \lambda$ *and* $g = h_\lambda^r$ *for some* $r \in \mathbb{Z}$.

We know that $g = \lambda g_1 \mu^{-1}$, $a = \lambda h_\lambda^s \lambda^{-1}$ and $b = \mu h_\mu^t \mu^{-1}$, and we also have that $a = g\delta(b)g^{-1}$. It follows that $g_1 \delta(h_\mu^t) g_1^{-1} = h_\lambda^s$

Observe that $h_\lambda^s = B_\lambda C_\lambda \dots B_\lambda C_\lambda$ is a reduced word for h_λ^s. Moreover, $h_\lambda^s \in \Gamma_1$ has minimal length among its conjugates, in the amalgamated free product decomposition of Γ_1. The same is true for $\delta(h_\mu)$. It follows that $s = \pm t$. We can assume that g_1 has minimal length among the elements $h_\lambda^{r_1} g_1 \delta(h_\mu)^{r_2}$ with $r_1, r_2 \in \mathbb{Z}$. With this assumption, it also follows that $g_1 \in \{e, C_\lambda^{-1}, B_\lambda\} G_0 \{e, \delta(C_\mu), \delta(B_\mu)^{-1}\}$.

In any case, it follows that there are $g_2, g_3 \in G_0$ such that $g_2 \delta(B_\mu)^{\sigma_2} = B_\lambda^{\sigma_3} g_3$ where $\sigma_2, \sigma_3 = \pm 1$. Moreover, we get that $g_1 = b g_2 c$ where $b \in \{e, C_\lambda^{-1}, B_\lambda\}$ and $c \in \{e, \delta(C_\lambda), \delta(B_\lambda)^{-1}\}$. We also see that $\sigma_2\sigma_3 = -1$ if and only if either b or c is e, but not both.

Using the fact that $\delta = \mathrm{Ad}_E * \mathrm{Ad}_F$ with E, F as in step 1, we see that $g_2 E B_\mu^{\sigma_2} = B_\lambda^{\sigma_3} g_3 E$. We write g_2 and g_3 as block matrices

$$g_2 = \begin{pmatrix} AXA^{-1} & 0 \\ 0 & 1 \end{pmatrix} \text{ and } g_3 = \begin{pmatrix} AYA^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

where X and Y are 2×2 matrices with integer coefficients that are upper triangular with eigenvalues 1, modulo 6, and that are lowe triangular with arbitrary eigenvalues modulo 7. Denote $\tau^{\sigma_2}(x_\mu)$ for x_μ if $\sigma_2 = 1$ and $\overline{x}_\mu$ if $\sigma_2 = -1$. We use also the similar notations with σ_3 and x_λ. Then we find that

$$AXVA^{-1}\begin{pmatrix} \overline{\tau^{\sigma_2}(x_\mu)} & 0 \\ 0 & 0 \end{pmatrix} = \sigma_1 \begin{pmatrix} \overline{\tau^{\sigma_3}(x_\lambda)} & 0 \\ 0 & 0 \end{pmatrix} AYVA^{-1} \tag{1}$$

$$AXVA^{-1}\begin{pmatrix} n_\mu \\ 0 \end{pmatrix} = \sigma_1\sigma_2\sigma_3 \begin{pmatrix} n_\lambda \\ 0 \end{pmatrix} u \tag{2}$$

$$u\begin{pmatrix} n_\mu \\ 0 \end{pmatrix} = \sigma_1\sigma_2\sigma_3 \begin{pmatrix} n_\lambda \\ 0 \end{pmatrix} AYVA^{-1} \tag{3}$$

$$\text{and } u\tau^{\sigma_2}(x_\mu) = \sigma_1 \tau^{\sigma_3}(x_\lambda) u \tag{4}$$

where V and u are as in the definition of E in step 1, and $\sigma_1 = \pm 1$. The equation (4) implies that $|x_\mu| = |x_\lambda|$, so it follows that $\mu = \lambda$. Moreover, comparing the real parts in (4), we see that the sign $\sigma_1 = 1$.

From equation (2), we conclude that

$$\xi = A^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} h \\ -i\overline{h} \end{pmatrix}$$

is an eigenvector of XV with eigenvalue $x = \sigma_2\sigma_3 u$.

We have to take a little care with what we mean by eigenvalues and eigenvectors over non-abelian rings, but the definition is exactely as in (2): the matrix is on the left of the vector, while the scalar is on the right. In a skew field, like $\mathbb{H}$, we say that two elements x, y are equivalent if they are conjugates of each other. Observe that in $\mathbb{H}$ two elements are equivalent if and only if they have the same modulus and the same real part. For the vector spaces $\mathbb{H}^n$, we always consider scalar multiplication on the right and matrix multiplication on the left. If Z is a square matrix over $\mathbb{H}$ and η is an eigenvector of Z with eigenvalue $y \in \mathbb{H}$. Then we see that a scalar multiple ξz is still an eigenvector of Z but with eigenvalue $z^{-1}yz$.

Now we see that the vector $(1, ih)$ is an eigenvector of XV with eigenvalue $\tilde{u} = \sigma_2\sigma_3 hu\overline{h}$. Observe that $\tilde{u}$ is still in $\mathcal{U}$. Write X and V as matrices

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } V = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \text{ or } V = \begin{pmatrix} vk & 0 \\ 0 & -vk \end{pmatrix},$$

where $v \in \mathcal{U} \cap \mathbb{C}$ and a, b, c, d are integers. Because $(1, ih)$ is an eigenvector, we see that $cv + dvih = ih(av + bvih)$ or $cvk - dvkih = ih(avk - bvkih)$. Observe that v, vih, ihv and $ihvih$ are linearly independent over $\mathbb{R}$ unless $v = \pm 1$. We know that $X \neq 0$, so we conclude that $v = \pm 1$. Similarly, $vk, vkih, ihvk$ and $ihvkih$ are linearly independent over $\mathbb{R}$ unless $v = \pm\frac{1-i}{\sqrt{2}}$.

We check both cases separately. Suppose we are in the first case, with $v = \pm 1$.One easily computes that $c = -b$ and $a = d + b$. Since $X \in G_0$, we see that b is a multiple of 6 and d is congruent to 1 modulo 6. Remark that $\tilde{u} = av + bvih$, while b is a multiple of 6 and a is congruent to 1 modulo 6. The only element in $\mathcal{U}$ that can be written in this form is v itself, so we conclude that $\tilde{u} = v$, i.e $u = \sigma_2\sigma_3 v = \pm 1$. But if $\sigma_2\sigma_3 = -1$, then equation (4) tells us that $x_\lambda = \overline{x_\lambda}$, which is simply not the case. So $u = v$ and hence $\delta = \mathrm{id}$. Moreover, we see that $X = 1$ and so $g_1 = 1$. So in the first case, we are done.

Assume now that we are in the second case and $v = \pm\frac{1-i}{\sqrt{2}}$. A direct computation shows that $a = d$ and $c = -b - a$. Moreover, $a, -b, d$ are congruent to 1 modulo 6 while c is a multiple of 6, and b is a multiple of 7. Since $avk - bvkih = \tilde{u} \in \mathcal{U}$, we see that this is not possible. This contradiction shows that we must have been in the first case, and the lemma is proven. □

6. Examples of endomorphism semigroups

In this section, we use the results of sections 4 and 5 in order to show that many semigroups appear as $\mathrm{End}(M)$ for some type II$_1$ factor M. The starting point is the following.

Theorem 6.1. *Let $\Lambda \curvearrowright (Y, \nu)$ be any probability measure preserving action of a not necessarily discrete group. Then there is a type II_1 factor M such that*

$$\mathrm{End}(M) \cong \mathrm{Factor}(\Lambda \curvearrowright (Y, \nu))^{\mathrm{op}}$$
$$\text{and } \mathrm{RFBimod}(M) \cong \{\text{formal finite direct sums of elements of } \mathrm{Factor}(\Lambda \curvearrowright (Y, \nu))^{\mathrm{op}}\}$$

Proof. First of all, we can assume that $\Lambda \subset \mathrm{Aut}_\nu(Y)$. Take any countable dense subgroup Λ_0 of Λ. We observe that $\mathrm{Factor}(\Lambda \curvearrowright Y) = \mathrm{Factor}(\Lambda_0 \curvearrowright Y)$. So we can assume that Λ is a countable group.

Then this theorem is a direct consequence of our main result 4.1, the flexible class of examples in section 5, and of lemma 6.3 below. □

Before we prove lemma 6.3, we show that many left cancellative semigroups appear this way. First, observe that any *compact* left cancellative semigroup with unit is in fact a group. But then we get that $G = \mathrm{Factor}(G \curvearrowright (G, h))$ where h denotes the Haar measure on G. So the compact case is easy to handle, but nor very interesting.

A more interesting class of left cancellative semigroups is the class of discrete left cancellative semigroups. This class contains proper semigroups, like $\mathbb{N}$, and even semigroups that can not be embedded into groups (for example they are not right cancellative). We show that any discrete left cancellative semigroup appears as the (opposite of the) semigroup of factors of some probability measure preserving action.

Let G be a left cancellative semigroup. Let (Y_0, ν_0) be a standard nonatomic probability space and consider $(Y, \nu) = (Y_0, \nu_0)^G$. Every element $g \in G$ defines a measure preserving quotient map $f_g : Y \to Y$ by the formula $f_g(x)_h = x_{gh}$. Then we see that $f_g f_h = f_{hg}$. So we see that $G^{\mathrm{op}} \subset \mathrm{Factor}(Y, \nu)$. If we take a non-atomic base space (Y_0, ν_0), and we choose an appropriate subgroup $\Lambda \subset \mathrm{Aut}_\nu(Y)$, then we get that $G^{op} = \mathrm{Factor}(Y, \nu)$:

Lemma 6.2. *Let G be a left cancellative semigroup with unit e, and let (Y_0, ν_0) be a non-atomic probability space. Set $(Y, \nu) = (Y_0, \nu_0)^G$ as before. Then there is a subgroup $\Lambda \subset \mathrm{Aut}_\nu(Y)$ such that*

$$\mathrm{Factor}(Y, \nu) = G^{op}.$$

Proof. Denote by $\mathcal{G} \subset \mathrm{Aut}_\nu(Y)$ the closed subgroup of $\mathrm{Aut}_\nu(Y)$ that is generated by the following transformations:

$$\psi_\Delta : Y \to Y \quad \psi_\Delta(x)_h = \Delta(x_h) \qquad\qquad \text{for all } \Delta \in \mathrm{Aut}_{\nu_0}(Y_0)$$

$$\varphi_{U,g,\Delta} : Y \to Y \quad \psi_\Delta(x)_h = \begin{cases} \Delta(x_h) & \text{if } x_{hg} \in U \\ x_h & \text{if } x_{hg} \notin U \end{cases} \quad \begin{array}{l} \text{for all } \Delta \in \mathrm{Aut}_{\nu_0}(Y_0) \text{ with } \Delta(U) = U \\ \text{and for all } g \in G, U \subset Y_0 \end{array}$$

These automorphisms commute with all the f_g with $g \in G$. In other words, we see that $G^{\mathrm{op}} \subset \mathrm{Factor}(\mathcal{G} \curvearrowright Y)$. We show that this is actually an equality. Let $f : Y \to Y$ be a measure preserving quotient map that commutes with the action of $\mathcal{G}$.

step 1: *Because f commutes with all the ψ_Δ, there is an injective map $\alpha : G \to G$ such that $f(x)_g = x_{\alpha(g)}$ for almost all $x \in Y$, and for all $g \in G$.*

Fix a set $U \subset Y_0$ with $0 < \nu_0(U) < 1$, and consider the group $H = \{\Delta \in \mathrm{Aut}_{\nu_0}(Y_0) \mid \Delta(U) = U\}$. Then we know that H acts weakly mixingly on both U and $Y_0 - U$. In particular, for

every finite set I, we know that the H-invariant subsets of $(Y_0, \nu_0)^I$ are precisely the disjoint unions of sets of the form $U^{I_1} \times (Y_0 - U)^{I - I_1}$ for subsets $I_1 \subset I$.

Fix $i_0 \in G$ and denote by $f_{i_0} : Y \to Y_0$ the composition of f with the quotient map onto the i_0-component. Write $\widetilde{U} = f_{i_0}^{-1}(U)$. For a fixed finite subset $I \subset G$ and an element $y \in Y_0^I$, denote

$$\begin{aligned} \widetilde{U}_{I,y} &= \left\{ x \in Y_0^{G - I} \;\middle|\; f(y,x)_{i_0} \in U \right\} \\ \widetilde{U}_I &= \left\{ y \in Y_0^I \;\middle|\; \nu_0^{G - I}(\widetilde{U}_{I,y}) > 0 \right\}. \end{aligned}$$

Now it is clear that $\widetilde{U} \subset \widetilde{U}_I \times Y_0^{G - I}$. Moreover, $\widetilde{U}_I$ is a disjoint union of sets of the form $U^{I_1} \times (Y_0 - U)^{I - I_1}$ for subsets $I_1 \subset I$.

We claim that there is an $i \in I$ such that $U^{\{i\}} \times Y_0^{I \setminus \{i\}} \subset \widetilde{U}_I$. To prove this claim, put $n = |I|$ and observe that the function $p \mapsto (1-p)^{n-1} + np(1-p)^{n-2}$ tends to 1 as $p \to 0$, but the derivative in 0 is positive. So, taking p small enough, we can assume that $(1-p)^p + np(1-p)^{n-1} > (1-p)$. Take a subset $V \subset U$ with measure $\nu_0(V) = p$. Write $\widetilde{V} = f_{i_0}^{-1}(V)$ and define $\widetilde{V}_I$ in the same way as we defined $\widetilde{U}_I$. Then it is clear that $\widetilde{V}_I \subset \widetilde{U}_I$, but also that $\nu_0^I(\widetilde{V}_I) \geq p$. Observe that $\widetilde{V}_I$ is a disjoint union of sets of the form $V^{I_1} \times (Y_0 - V)^{I - I_1}$ for some subset $I_1 \subset I$. It follows that $\widetilde{V}_I$ must contain at least one of the sets $(Y_0 - V)^I$ or $(Y_0 - V)^{I - \{i\}} \times V^{\{i\}}$ for some $i \in I$. In the first case, it follows that $Y_0^I = \widetilde{U}_I$, while in the second case we find that $Y_0^{I - \{i\}} \times U^{\{i\}} \subset \widetilde{U}_I$. In both cases, we have proven our claim.

Observe that, for any $\varepsilon > 0$, there is a finite set $I \subset G$ such that $\nu(\widetilde{U}_I \times Y_0^{G - I} - \widetilde{U}) < \varepsilon$. Taking $\varepsilon = \nu_0(U)(1 - \nu_0(U))$, we find a finite subset $I_0 \subset G$ such that $\nu(\widetilde{U}_I \times Y_0^{G - I} - \widetilde{U}) < \varepsilon$. Then there is a unique $i \in I_0$ with $Y_0^{I_0 - \{i\}} \times U^{\{i\}} \subset \widetilde{U}_{I_0}$. For every finite set $I_0 \subset I \subset G$, we see that $Y_0^{I - \{i\}} \times U^{\{i\}} \subset \widetilde{U}_I$, with the same $i \in I_0 \subset I$. As a consequence we find that $Y_0^{G - \{i\}} \times U^{\{i\}} \subset \widetilde{U}$. Comparing the measures, we see that this inclusion is actually an equality. In other words, $f_{i_0}^{-1}(U) = U^{\{i\}}$.

Every subset $V \subset U$ is of the form $V = U \cap \Delta(U)$ for some automorphism $\Delta : Y_0 \to Y_0$. Hence we compute that

$$\begin{aligned} f_{i_0}^{-1}(V) &= f^{-1}(U) \cap \psi_\Delta(f_{i_0}^{-1}(U)) \\ &= U^{\{i\}} \cap \Delta(U)^{\{i\}} \\ &= V^{\{i\}} \end{aligned}$$

The same proof works for subsets of $Y_0 - U$. Hence we get that for every $V \subset Y_0$, the inverse image $f_{i_0}^{-1}(V)$ equals $V^{\{i\}}$. In other words, $f(x)_{i_0} = x_i$ almost everywhere.

We can do this for every $i_0 \in G$ and find a map $\alpha : G \to G$ such that $f(x)_g = x_{\alpha(g)}$. The map α is injective because otherwise f would not be measure preserving. This finishes the proof of step 1.

step 2: *Because f also commutes with the $\varphi_{U,g,\Delta}$, it follows that $f = f_k$ for some $k \in G$.*

In fact, we only need that f commutes with $\varphi_{U,g,\Delta}$ for one non-trivial set $U \subset Y_0$ and one non-trivial automorphism $\Delta : Y_0 \to Y_0$, but for all $g \in G$. Remember that f is of the form $f(x)_h = x_{\alpha(h)}$ almost everywhere and for all $h \in G$. The fact that f commutes with $\varphi_{U,g,\Delta}$ implies that $\alpha(hg) = \alpha(h)g$. Set $k = \alpha(e)$ where e is the unit of g. Then it follows that $\alpha(g) = \alpha(e)g = kg$. In other words, we see that $f = f_k$. This concludes the proof of step 2, and hence the proof of lemma 6.2. □

Finally we prove the technical lemma we needed in the proof of theorem 6.1.

Lemma 6.3. *Let $\Lambda \curvearrowright (Y,\nu)$ be an action of a countable group on a probability space (Y,ν), then there is a probability measure preserving action $\widetilde{\Lambda} \curvearrowright (\widetilde{Y},\tilde{\nu})$ of an anti-(T) group $\widetilde{\Lambda}$ (see definition 3.1) such that*

$$\mathrm{Factor}(\widetilde{\Lambda} \curvearrowright (\widetilde{Y},\tilde{\nu})) = \mathrm{Factor}(\Lambda \curvearrowright (Y,\nu))$$

and such that this new action does not have any non-trivial cocycles to compact groups. Moreover, the new action is ergodic.

Proof. We use the generalized co-induced actions that were introduced in [D]. For the convenience of the reader, we repeated the construction and basic properties in preliminary 1.2.

Without loss of generality, we can assume that $\Lambda = \mathbb{F}_\infty$. Denote by a_n the n-th canonical generator. Consider now the following group:

$$\widetilde{\Lambda} = \underbrace{\mathrm{SL}_2\mathbb{Q} \ltimes \mathbb{Q}^2}_{H} *_\Sigma \underbrace{(\underset{G_0}{\underset{\shortparallel}{\mathrm{SL}_2\mathbb{Z}}} \ltimes (\underset{A}{\underset{\shortparallel}{\mathbb{Z}^2}} \times (\underset{B}{\underset{\shortparallel}{\mathbb{Z}^2}} * \underset{F_1}{\underset{\shortparallel}{\mathbb{Z}^2}} * \underset{F_2}{\underset{\shortparallel}{\mathbb{Z}^2}} * \ldots)))}_{G},$$

$$\begin{aligned} \text{where } \Sigma &= \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\} \subset G \\ &\cong \left\{ \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^n \mid n \in \mathbb{Z} \right\} \subset H. \end{aligned}$$

Observe that $\widetilde{\Lambda}$ is an anti-(T) group because it is an amalgamated free product of poly-Haagerup groups. There is an obvious quotient from $\widetilde{\Lambda}$ onto the group

$$\widetilde{\Lambda}_1 = H *_\Sigma (G_0 \ltimes (A \times B)),$$

where the F_n are mapped to the identity element in $\widetilde{\Lambda}_1$. Consider the group $\widetilde{\Lambda}_0 = \mathrm{ST}_2\mathbb{Q}$ of upper triangular matrices in $\mathrm{SL}_2\mathbb{Q}$, and let $\widetilde{\Lambda}$ act on $I = \widetilde{\Lambda}_1/\widetilde{\Lambda}_0$ by left translation. Define a cocyle $\omega : \widetilde{\Lambda} \times I \to \mathbb{F}_\infty$ by the following relations.

$$\begin{aligned} \omega(s,i) &= e && \text{if } s \in \widetilde{\Lambda}_0 \\ \omega(s,i) &= e && \text{if } s \in F_n \text{ and a reduced word for } i \text{ starts with a letter from } H \\ \omega(s,i) &= a_n^{\det(s|b)} && \begin{array}{l}\text{if } s \in F_n \text{ and} \\ (a,b)g \in G_0 \ltimes (A \times B) \text{ is the first letter of a reduced word for } i.\end{array} \end{aligned}$$

Above, we denoted $\det(s|b)$ for the determinant of the matrix whose columns are s and b.

Consider the generalized co-induced action $\widetilde{\Lambda} \curvearrowright \widetilde{Y}$ of $\Lambda \curvearrowright Y$, associated to the cocycle ω. This cocycle clearly satisfies the conditions of lemma 1.4. So we see already that

$$\mathrm{Factor}(\widetilde{\Lambda} \curvearrowright (\widetilde{Y}, \tilde{\nu})) = \mathrm{Factor}(\Lambda \curvearrowright (Y, \nu)),$$

and moreover that $\widetilde{\Lambda}$ acts ergodically on $\widetilde{Y}$.

It remains to show that every cocycle $\alpha : \widetilde{\Lambda} \times \widetilde{Y} \to \mathcal{G}$, to a compact group $\mathcal{G}$, is in fact trivial. When restricted to $\widetilde{\Lambda}_0$, the action is just a generalized Bernoulli action. We can apply Popa's cocycle superrigidity theorem [P4] to the relatively rigid inclusion $\mathbb{Z}^2 \subset \mathbb{Q}^2 \subset H$. We find a measurable map $\varphi : \widetilde{Y} \to \mathcal{G}$ such that $\varphi(gx)\alpha(g,x)\varphi(x)^{-1}$ is independent of the x-variable for every $g \in \mathbb{Z}^2$. Since $\mathbb{Z}^2$ is almost normal in H and acts weakly mixingly on $\widetilde{Y}$, the same is in fact true for all $g \in H$.

Analogously, using the rigid inclusion $A \subset G$, we find a measurable function $\varphi_2 : \widetilde{Y} \to \mathcal{G}$ such that $\varphi_2(gx)\alpha(g,x)\varphi_2(x)^{-1}$ is independent of the x-variable for all $g \in G$. For $g \in \Sigma$, both

$$\theta(g) = \varphi(gx)\alpha(g,x)\varphi(x)^{-1} \text{ and } \theta_2(g) = \varphi_2(gx)\alpha(g,x)\varphi_2(x)^{-1}$$

are independent of x. So we find that

$$(\varphi\varphi_2^{-1})(gx) = \theta(g)(\varphi\varphi_2^{-1})(x)\theta_2(g)^{-1}$$

for all $g \in \Sigma$. Since Σ acts weakly mixingly on $\widetilde{Y}$, it follows that $\varphi = \varphi_2$ up to a constant. So we can assume that φ actually equals φ_2, and α is cohomologuous to a group morphism $\theta : \widetilde{\Lambda} \to \mathcal{G}$.

Since the only group morphism $H \to \mathcal{G}$ is the trivial morphism, we see that at least $H \subset \ker\theta$. In particular, $\Sigma \subset \ker\theta$. The smallest normal subgroup of G that contains Σ is G itself, so it follows that $G \subset \ker\theta$. But then θ is the trivial group morphism. □

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